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PRACTICAL METHODS IN ARITHMETIC

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BY

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D. C. HEATH & CO., PUBLISHERS
BOSTON NEW YORK CHICAGO

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PRACTICAL METHODS IN ARITHMETIC

FIRST PART — INTRODUCTORY

CHAPTER I

PLAN AND SCOPE

The following chapters suggest a working procedure that may be followed in the teaching of the several topics that properly belong to the mathematical course of the eight years of the elementary school. The plan of the work precludes the presentation of a multiplicity of devices in teaching a given topic to the pupils of a certain grade; it does, however, suggest the manner in which each subject may profitably be treated throughout the grades in which it is usually taught.

Believing that some of the unsatisfactory results in arithmetic are due to over-zealous teachers who think that there can be no merit in doing things in the direct and obvious way, and that children derive some peculiar benefit by being required to perform their tasks in a manner that is considered pedagogical because it is roundabout and difficult, the following pages are intended to show the simplest way of teaching arithmetic, and that the simplest way is the most scientific one. Although the suggestions are presented in plain language, carefully avoiding the employment of the technical terminology of pedagogy and of psychology, the opinions of the authorities on mathematical teaching are carefully followed.

The young teacher does not always make a distinction between the instruction in the subject matter he receives in the normal school, and the method he is taught to follow in teaching it in the elementary school. In order that he may be well grounded in arithmetic, he receives special instruction in its science; but he sometimes fails to separate this course in arithmetic from the accompanying one in methods. In this way there has arisen a tendency to inflict upon young children an explanation of the reasons underlying the different processes, instead of drilling them in the art of manipulating numbers. This book lays particular emphasis upon the futility of all such attempts, and the consequent waste of time. In the methods suggested for the early lessons the object is to enable the pupils to become proficient in addition by adding, and not by talking about it. When the development of the pupil is such as to enable him to understand the reason for each step in a process, the method suggested is intended to develop this reason.

Although all of the topics found in the ordinary textbook receive careful attention, the omission of many of them is suggested to such teachers as are given any discretion in the matter. The limitation of the details of many other topics is also advised. The tendency to overload the elementary course is always with us. The fact that some topic or some detail is simple is considered sufficient reason for finding a place for it. Topics properly belonging to upper grades are simplified, and unnecessarily thrust upon the children a few years too soon.

The grade teacher adds to the pupils' burdens by his failure to appreciate the limitations of the course; and to avoid criticism that he thinks might otherwise follow, he introduces many details that belong to a higher grade. It is hoped to show the readers of this book that while a child is benefited by work up to the limit of his capacity, he is prevented from learning essential things by attempts to give him tasks beyond his ability to perform properly.

No originality is claimed for the book. The aim is merely to outline a workable plan which the thoughtful teacher can modify and adapt to his own peculiar conditions.

The numerous repetitions in arrangement that appear, might be avoided by references to other portions of the text, but it is believed that there is an advantage in repeating a suggestion when the proper occasion offers, rather than to interrupt the reading with a foot-note referring to the page containing the original statement.

The teacher must keep constantly in mind that two methods of teaching the same topic may differ materially without either being wrong. He should understand that those given are merely suggestive, that they may not be applicable in a particular case, and that they may need modification to suit certain conditions.

The methods recommended are not the only good ones, nor are they given in great variety. The thoughtful teacher in his reading will discover many others, some of which may seem better suited to his case. Those presented are intended to outline a logical course of procedure for the four years of the primary course, as being more useful to a young teacher than a collection of detached methods of teaching scattered types of examples belonging to different parts of the course.

In suggesting methods, the author has chiefly in mind the teacher who has not more than two grades in his room. When all of the members of the class are of the same grade, they should generally be taught in two sections. This will be discussed in another chapter.

For convenience the work is divided into four parts. The first is general in character, the second is devoted to the work of the first four years, the third to the work of the fifth and sixth years, and the fourth to the work of the seventh and eighth years.

CHAPTER II

THE COURSE OF STUDY

THE arrangement of the text-book of a generation ago followed the logical development of the subject matter. An arithmetic began with definitions of mathematics, quantity, number, etc. Then came a chapter on numeration and notation of whole numbers, followed by successive chapters on addition, subtraction, multiplication, etc., each containing its quota of definitions, principles, and rules.

When the series consisted of two books, the lower volume was merely a condensation of the higher, or complete arithmetic, and was, if possible, more difficult because of the smaller number of exercises. Its arrangement of topics and its manner of treating them followed the plan of the larger book.

The too-conscientious teacher, not appreciating the fact that a text-book is made to serve many purposes, felt it his duty to teach everything contained in the book and to complete each page before beginning the next. Not knowing that the definitions, etc., were presented for the benefit of persons studying without a teacher, or of parents desirous of aiding their children, he insisted that all definitions, rules, and explanations should be memorized. Not realizing that the topical arrangement is essential in a book that is to be used as a work of reference by persons reviewing their studies, and the like, he taught each topic to the extent it was found in the book.

The older courses of study followed this procedure of the teachers, making processes the basis for the yearly divisions, the first year being devoted to addition, the second to subtraction, the third to multiplication, the fourth to division, the fifth to fractions, the sixth to decimals and denominate numbers, the seventh to percentage and interest, and the eighth to mensuration and any other arithmetical topics that had not been previously treated.

Although this system was not without its merits, the more thoughtful teachers felt that there was a better way. They saw that children were required to work tedious and difficult examples under one process before being permitted to try the simple ones of a later topic. Various modifications of the course of study for beginners were experimented with, the one receiving the greatest recognition being known as the Grube system.

The Grube system laid chief stress upon the study of numbers, the study of processes being made entirely subordinate. The child began with the number 1, and spent a year in completing the analysis of the first nine numbers, studying each in every possible way, before proceeding to the next. Using objects with each, the pupil was led to discover in the study of 2, for instance, that it is equal to 1 and 1, that 2 less 2 leaves 0, that 2 less 1 is 1, that 2 is two 1's, that 2 contains 2 once, that 1 is one-half of 2, etc., etc.

Problems involving the foregoing facts were solved by the pupil, and the work under 2 was completed when the pupil knew all of the foregoing combinations with abstract numbers. The exhaustive treatment of each number, which was considered an essential feature of the system, required that each successive number should receive more time than the preceding one, two months being considered about sufficient for the number 9.

Numbers to 100 constituted the second year's work, a similar treatment, modified in its details, being given to each. Numbers to 1000 constituted the course for the third year, work in all four of the fundamental operations being carried on simultaneously during these three years.

The Grube system in its entirety obtained no firm foothold in this country. A modified form was employed by many schools for the oral work of the earlier years, the mental problems of each year including addition, subtraction, multiplication, and division combinations.

In the written work, however, each operation was introduced at successive periods. The oral work, from almost the outset, was made to develop some knowledge of simple fractions and the commoner denominate units.

The unsatisfactory results that followed the employment of a course of study of this character were due to the overloading indulged in by its injudicious advocates. Considering themselves progressive, they introduced into the work of each year a number of details, each simple enough of itself, but tending by their multiplicity to prevent the young mind from acquiring definite knowledge of any one of them.

While the ideal course in arithmetic might be one that would be developed by the need of the pupils to express themselves in number and to perform the numerical operations growing out of their various activities, such a course has not as yet been formulated. From time to time attempts to find it have been made in "model" schools, and not without some measure of success; but pupils of the average school, subject to average conditions, must pursue their studies along lines calculated to produce the most permanent results.

Bearing in mind all of the conditions, the most thoughtful educators favor a course of study that is much simpler than many now in use. They are chiefly concerned with the results of a pupil's training at the end of four, six, and eight years, believing that it is of more importance for him to have a working knowledge of essential things when he leaves school than a useless familiarity with a number of things during the earlier years. They feel that the results in arithmetic now obtained, do not justify the expenditure of the time at present given to this branch, which, they think, should be diminished rather than increased. Their course would distinguish carefully between the necessary and the cultural, requiring that the latter be not pursued at the expense of the former. Pupils of the lower grades readily learn the facts of number, and the few added facts essential in further calculations should be mastered during early school life. Every child during the first four years should learn to add, subtract, multiply, and divide whole numbers, to add two or three mixed numbers containing the ordinary business fractions, to multiply by a mixed number, to have a working knowledge of decimals to the extent of dollars and cents, and to be acquainted with the more commonly used denominate units.

The ability to perform the fundamental operations is best secured by abundant practice in working a great many examples containing small numbers, combined with the necessary drill in the tables. All time spent in the endeavor to learn the reasons underlying the processes is wasted, as is the time devoted to the memorizing of definitions, rules, and the like. There should be a definite reason for the introduction of each form of weighing, measuring, paper folding, etc. The time at which each should be commenced should be determined, as well as the time for its discontinuance.

Objects are used in number work to enable a pupil to obtain results without them; and the longer their employment continues after the need has passed, the greater the possibility of injury to the pupil, aside from the loss of time. The same is true of unnecessary drawing to prove that 6 times 5 gives the same product as 5 times 6. The pupil who needs to divide a rectangle to learn this law as regards 2 × 3 and 3 × 2, should not require the picture for all other possible combinations. The same is true of fractions. Every reduction should not require paper folding. When the pupil knows that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$ are equal, he does not need to verify the equality of $\frac{8}{24}$ and $\frac{1}{3}$ by drawing a circle and dividing it up into the necessary parts.

The premature introduction of simplified forms of advanced work is responsible for many of these time-consuming devices. The pupil of the ungraded school is fortunate in not being likely to have his mental development arrested by tasks of this character.

During the fifth and sixth years, the courses should provide an adequate treatment of fractions, common, decimal, and compound numbers, with a preliminary treatment of percentage and measurements.

The course for the seventh and eighth years should complete the ordinary topics in arithmetic. Time for simple work in equational arithmetic and geometrical drawing might be obtained by omitting stocks and bonds, longitude and time, and other unnecessary topics.

The best practical courses now limit the oral number work of the beginners to problems in addition, the first lessons being the addition of 1 to successive numbers below 9, which is accompanied by adding to 1 the numbers below 9. The next step consists in solving problems involving the addition of 2 to numbers below 8, and the addition of these numbers to 2. In the remaining steps sums not greater than 9 are taught, first increasing by 3 the numbers from 3 to 6 and the reverse, then increasing 4 and 5 by 4. This follows the Grube system in dealing at the outset with numbers to 9, but it employs only one operation, addition. The next stage extends oral addition to the finding of the sum of any two digits. With this is introduced oral subtraction within the limits of the beginning addition work, the earlier minuends being limited to numbers not greater than 9.

Written work is now begun with the addition of numbers of two figures, the total of each column not exceeding 9. During the next term the written addition includes examples in which the column totals may extend to 19, for instance; and written subtraction is commenced, each figure in the minuend of every example being larger than the corresponding figure of the subtrahend.

The subtraction work of the following term requires the finding of the difference of any two numbers, and written multiplication is introduced, the latter being limited to some extent. Division is generally taken up with multiplication, although in some cases its introduction is postponed for a term.

Work in fractions is generally pursued in connection with division, the finding of the half of a number being required after pupils have learned to divide by 2. During the third year the addition of easy fractions is begun by finding the sum of two or three mixed numbers, the fraction in each being $\frac{1}{2}$. All of the fractions in each subsequent addition example during this term have like denominators, the latter being limited to 2, 4, and 3.

Decimals are first studied through examples in dollars and cents. Simple denominate numbers form a portion of the work of several grades, compound numbers being postponed, perhaps, until the fourth year.

To each grade is assigned its appropriate oral work, which includes the determination of the relation between the pint and the quart, between the peck and the bushel, between the inch and the foot, the pupils employing measures for this purpose. It includes also the drill work required in the special topic of the grade. Each grade should likewise have oral work intended to give proper drill in number combinations, beginning with the smallest numbers, the size of the latter increasing from grade to grade. Types are given under the different topics: addition, subtraction, etc.

In the instruction of a class, the teacher should not assume that the topics prescribed for his grade are to be taught in the order in which they are given. The course for a grade is frequently divided into two parts, one headed "Oral Work" and the other "Written Work." This by no means signifies that a portion of the year is to be given to the first, and the remainder to the second. The subdivisions under each part may be numeration and notation, addition, subtraction, etc., but it does not follow that the topics are to be taken up in the order in which they are specified.

The average course of study provides, together with the extension of old work, the introduction of new work. Even when they are not prescribed, it is expected that frequent reviews of completed topics will be had.

As a rule, the first lesson to a new class should deal with the special work of the grade. No matter how deficient he may be, the average pupil does not like to be told explicitly or impliedly that he has failed to master the lessons of the previous term. After preliminary drill exercises of a few minutes the first formal lesson after promotion should present to the pupils advanced work. There will be abundant opportunities for reviews, and these should be made subordinate, at least for a short time. All review lessons, when they are given, should generally consist of exercises not previously used. The teacher should provide them, when they are not found in the portion of the text-book that the class is studying.

The teacher should carefully refrain from making any additions to a prescribed course, which may possibly contain more than sufficient for the best interests of the pupils. The fact that a topic is found in that portion of a text-book devoted to a certain year, does not necessarily mean that it should be taught at that time, and the teacher should omit it when it is not specified in the course of study.

The average text-book, being intended for use in schools of every type in all sections of the country, contains many problems that should be omitted in a particular school.

All problems in the earlier years should be based upon the experience of the child. Different communities have different industries and different modes of life, with which the teacher should familiarize himself just as thoroughly as he does with his text-books. In an agricultural section many problems should be based upon the terms of the farm. The chief industry of a section should furnish the basis for many exercises. This requires that the teacher should supplement the text-book by problems founded upon the experiences of the members of his class, whose cooperation should be obtained in supplying prices, conditions, etc. Their outlook should not, however, be narrowed, especially in the case of the older pupils, by being permitted to believe that prices, wages, conditions, etc., given in text-books are necessarily incorrect because they differ from those of their immediate vicinity, and that they must necessarily limit all their calculations to such as are required in the solution of local problems.

The teacher should not feel bound to use every example under a particular topic before proceeding to a new one, nor should he consider it necessary for all pupils to work the same number of examples. As a rule, he will find it necessary to supply a larger number of simpler types than are contained in the book, and he should feel at liberty to omit some of the more difficult ones or to postpone them to a later date.

Other suggestions will be given in the chapters on addition, subtraction, etc. For convenience of reference, the methods suggested for each topic are classified under the foregoing heads; although for some purposes the arrangement by grades, giving the topics studied in each with the methods appropriate to the particular grade, might be preferable. The same result is, however, obtained by discussing under each topic the time when its introduction is profitable to the child, and the method that should be employed in the different grades throughout which special instruction in the topic is given.

CHAPTER III

FORMS OF EXERCISES

For convenience, arithmetical exercises are roughly divided into *oral* or *written*, the answers to the former being obtainable without the aid of a pencil; while in the latter, the pencil is employed to write out the successive steps. The distinction is entirely arbitrary, since many exercises designated as "written" may be worked without a pencil by some pupils, and other pupils need a pencil to solve some problems designated in their books as "oral."

The answers to oral exercises are frequently written by all the pupils, in order that the teacher may ascertain how many are able to obtain the correct result. In such cases the pencils are taken only when the pupils receive the order from the teacher as is explained in the next chapter.

Oral exercises may be given orally by the teacher, or the numbers may be in the view of the pupils. In the latter case they are called *sight exercises*, the answers to which may be given orally, or written promptly at a signal.

In order to accustom pupils to omit superfluous figures, and also to combine numbers without rewriting them, exercises containing numbers larger than those employed in the sight exercises are placed on the blackboard. As in the case of the latter, only the answers are written; owing, however, to the size of the numbers contained in these so-called blackboard exercises, the pupils are not expected to obtain the complete result before taking the pencils. They write each figure when it is obtained, and are given a fair amount of time to obtain and set down the successive figures before receiving the order: "Pencils down."

In the lower grades, some exercises consist of the addition of numbers written on the board, the total of each column being announced orally by a pupil at his seat, while a pupil at the board writes each successive result. In other grades blackboard exercises are worked from the text-book. The following are types for appropriate grades below the fifth year:

(*a*) \$3.56 + \$.49 (*b*) \$4.05 — \$1.86
(*c*) 687 × 40 (*d*) 3 pounds of coffee @ 35¢
(*e*) $\frac{1}{3}$ of 2598 (*f*) 59 + ? = 400

In the grades above the fourth year, blackboard exercises are also employed to teach short methods.

The following are types of such methods:

(*a*) \$10 — (\$3.14 + \$.73 + \$4.85) (*b*) \$20 – 7 × \$2.84
(*c*) $\begin{array}{r} 9648 \\ \times\ 25 \\ \hline \end{array}$ (*d*) $\begin{array}{r} 2769 \\ \times\ 33\frac{1}{3} \\ \hline \end{array}$ (*e*) $\begin{array}{r} 1896 \\ \times\ .125 \\ \hline \end{array}$ (*f*) $\begin{array}{r} 346 \\ \times\ 99 \\ \hline \end{array}$
(*g*) $\frac{3}{8}$ of 483 (*h*) $\frac{2}{25}$ of 942 (*i*) $\frac{1}{99}$ of 1865

Other types are given in connection with the work of the different grades. Methods of using these various forms are found in the next chapter.

Exercises employed to develop facility in combining abstract numbers are called *drills*. These should constitute an important part of the work of the lower grades, each of which has its appropriate type. Drills of the previous grades are reviewed in each succeeding grade, as far as may be found necessary. Those peculiar to the several processes and to the various grades will be found in later chapters.

To be of value, quick oral answers in succession are essential in drill exercises, the rapidity of the replies increasing day by day until the answer to each combination flashes into the mind the moment it is seen by the eye.

A teacher having but one or two grades in his room should devote a few minutes each day to these drills. They may be placed on the blackboard, or the teacher may prepare a number of cards each containing a single combination, the figures being large enough to be distinctly visible to every pupil. As each new combination is taught, it may be added to those previously used.

Counting exercises constitute a form of oral drill in addition and multiplication. They will be discussed under those heads.

The early sight exercises are employed primarily to teach the combinations every child must know before he can perform the ordinary operations. There are forty-five of these in addition and thirty-six in multiplication, which should be thoroughly known. As a rule the early combinations should appear on the card or the blackboard in the form they assume in the child's written work:

$$\begin{array}{r} 8 \\ +\,9 \\ \hline \end{array} \qquad \begin{array}{r} 9 \\ \times\,5 \\ \hline \end{array} \qquad \begin{array}{r} 17 \\ -\,8 \\ \hline \end{array} \qquad \begin{array}{r} 2)\underline{16} \\ 8)\underline{16} \end{array} \qquad \begin{array}{r} 9 \\ +\,36 \\ \hline \end{array}$$

While the average child should give the answer as soon as he sees the combination, it will help some children to speak the combination, both the ear and the eye being needed by such pupils to make the necessary impression upon the mind. In order that the rapidity of the drill may not be interfered with, such pupils should constitute a separate section.

A form of counting exercise consists in the oral recitation of the tables by the class singly and in concert. Although concert recitations are tabooed, they are of use in memorizing poetry and the thirty-six combinations of the multiplication tables.

Other sight exercises are intended to teach number. They include larger combinations, and do not necessarily require the arrangement recommended for the others. Each may be written on a line with the proper sign between the two numbers, as, for instance,

(*a*) $60 + 30$ (*b*) $90 - 40$ (*c*) $80 \div 2$
(*d*) $80 \div 40$ (*e*) 3×30

The answers may be given orally by individuals or written by the class as a whole.

Arithmetical work, oral and written, is also divided into problems and abstract exercises. The latter are intended to teach the pupils to add, subtract, etc., the required operation being stated in connection with each example. In a problem the pupil is expected to determine from the given conditions what operation is required to obtain the answer.

Abstract written work under each process is preceded, accompanied, and followed by appropriate sight work. The time for the introduction of each, and the method employed in teaching it, with its changes from grade to grade, will be given in the proper chapters.

The main object in the teaching of arithmetic is to enable the boy or girl in later years to solve problems of various kinds. While these may take small range in the lives and occupations of most people, the aim of the teacher must be to develop as far as possible the ability of each member of his class. The first thing to be taught is a working familiarity with the tools, which is obtained in the abstract work. When this is secured, the child has made a step forward, even if his problem work is limited. This, to have any value, must keep pace with the pupil's development.

The immediate aim of problems in the lower grades is to enable the child to determine the necessary operation when he reads or hears the conditions. The first oral problems should be of such a character that he can determine the result at once. Those given in the early lessons consist of increasing a number by 1, later by 2, using objects in the class-room, etc. When he reaches written addition of three numbers of two figures each with no carrying involved, he may be asked to solve an oral problem in which the numbers to be added are such as 12 and 4. This is followed by a written one containing exactly the same conditions and expressed in the same terms, but with a change of the numbers to 22 and 4.

The problems of each grade should contain numbers just a shade more difficult than those of the oral work. When the combinations of the latter are such as (*a*) 30 + 50, (*b*) 80 ÷ 4, etc., the written problems should contain such as (*a*) 39 + 52, (*b*) 76 ÷ 4, etc., the conditions and wording being otherwise alike.

Until perhaps the second half of the third grade, each problem should involve but one operation, and the numbers, including those in the result, should not contain more than two figures. When problems are extended to include numbers of three figures, but one such number should be introduced at first. Thus, the oral problems involving the following combinations will contain three-figure answers: (*a*) 70 + 50, (*b*) 30 × 4; and the numbers in corresponding written ones may be (*a*) 79 + 58, (*b*) 37 × 4. The subtraction and division combinations will be, for the oral problems, (*a*) 130 — 70 and (*b*) 180 ÷ 6, while the written ones may contain (*a*) 130 — 79, and (*b*) 174 ÷ 6.

By careful grading the combinations of the oral problems of one class may be employed in the written ones of a lower class, care, however, being taken not to make them so simple that pupils will be required to use their pencils unnecessarily.

When numbers of three figures must be used in problems, the first oral combinations may be such as (*a*) 350 + 40, (*b*) 290 — 50, (*c*) 120 × 4, (*d*) 360 ÷ 3, etc., and the written ones, such as (*a*) 349 + 49, (*b*) 290 — 58, (*c*) 127 × 4, (*d*) 357 ÷ 3, etc.

The first problems of two operations should contain smaller numbers than those of one operation in the same grade, being limited as long as possible to numbers of two figures.

As problems are used to require the determination of the necessary operation, the conditions should be those within the experience and environment of the pupil. If a child is compelled to struggle with the meaning of the words or with unfamiliar conditions, he loses some of the attention required to enable him to determine the operations. Small numbers, simple language, familiar conditions are all important; moreover, they make it possible for the pupils to obtain the benefit obtained by solving many problems in the allotted time.

Some teachers assert that a child can just as readily solve a problem asking the number of plants in a 7-acre field, each acre containing 4840 plants, as he can one requiring the number of plants in 4 rows, each containing 15 plants; but experience shows the opposite. In the case of the smaller numbers the child's mind tends toward a mental solution. Pupils of the higher grades will find many different answers to a written problem containing large numbers, even though every member of the class will find the correct answer to the same problem given as an oral one with small numbers.

To be classified as a problem, an example should not state in terms that the given numbers are to be added, multiplied, etc. An example is not a real problem if it is placed under such a heading as Problems in Subtraction, Problems in Division, etc. An example of this kind is likely to cause a pupil to neglect to read it carefully, as he is already told the operation that is required. A set of problems should be miscellaneous, no two consecutive ones employing just the same conditions.

The number of written problems solved in a year should increase grade by grade. The problem work of the lower grades should not prevent the pupils from obtaining reasonable facility in the abstract work, since the latter is of chief importance. If the child solves comparatively few written problems during the first year or two he suffers no particular harm. The training received by the oral problems will enable him to solve reasonably well in the higher grades such written ones as are worked by his classmates.

When the pupils have text-books containing suitable problems, they should work the latter at their desks, where they can take the time necessary to understand the conditions.

Although a training in the science of arithmetic may be useful for disciplinary purposes, it is entirely out of place in the lower grades of the elementary schools, and much of it is adapted only to pupils that have reached the high school. Nevertheless, there are many teachers who fail to realize that they are wasting the time of their pupils in endeavoring to explain the reasons underlying the different processes. During the short time that many children are able to remain in school, the time devoted to their instruction in arithmetic should be given chiefly to making them reasonably expert in performing the fundamental processes with whole numbers, simple and compound, fractions, and decimals. Such problems as are given at each stage should be adapted to the age and the capacity of the children, and only after they are fairly well grounded in the abstract work.

CHAPTER IV

THE RECITATION

WHILE the teacher should be familiar with a subject in all of its details, his efficiency is frequently determined as much by what he withholds from his pupils as by what he tells. It is a commonplace that nothing should be told a child that he can readily determine for himself under proper direction, and the following methods suggest a maximum of talk by the pupils and a reasonable minimum by the teacher. The successful instructor states only the essential things, things that cannot be readily discovered by the learner; things that the latter is able to discover, the teacher elicits from him by successive questions that gradually lead up to the answer desired. This rule is to be followed intelligently. Some pupils will succeed in evading the most skilfully prepared questions. In such cases nothing is gained by wasting time; the needed explanation should be given.

Before beginning a recitation the teacher should be sure that the blackboard on which the main work is done, can be seen readily by every member of the class. Some pupils with impaired sight or hearing are frequently seated at the rear of the room, and they fail to obtain the full benefit of the recitation, thereby getting a reputation for stupidity even when they may be of more than average brightness. Some children do not wish to admit that they have defects, and, therefore, do not tell the teacher they are unable to hear his questions, etc.

In the lower grades each day's work should generally begin with a drill of a minute or two. The teacher may announce a counting drill. The class stands. The teacher gives the number to be added, for instance 4. He then says 1, and successive pupils in turn say rapidly 5, 9, 13, 17, 21, 25, etc., up to the limit set for the drill. The next pupil says 2, and those following him say, in turn, 6, 10, 14, 18, 22, etc. The next pupil says 3, and those following him say in turn, 7, 11, 15, 19, 23, etc.

Another form of this drill requires each pupil to give all the required sums, the first beginning with 1, and adding by 2's, for instance; the next beginning with 1 and adding by 3's; the third beginning with 2 and adding by 3's; the fourth beginning with 1 and adding by 4's, etc. To be of advantage, the successive numbers should be given as rapidly as possible.

Sight drills from the blackboard or from cards constitute another variety. The class stands, the teacher shows a card, the first pupil answers promptly, and then takes his seat. Another card is shown, the next pupil answers, etc. If a pupil hesitates or gives a wrong result, the teacher says "Class," and all the members give it. A pupil that has not given the correct answer walks to the teacher's desk, takes the card, and stands at one side of the room until the round of the class is made, which should require only a very short time. Those who have made mistakes arrange themselves before the teacher's desk, each hands in his card, and states the correct answer. These pupils then are put through the drill, each being required in turn to give an answer as a card is shown, the time for his reply being lengthened somewhat. Each of these pupils may be required to give the answers to several combinations.

When the drill is had by means of combinations previously placed upon the blackboard, pupils answer in turn, the teacher placing the pointer on the required combination, "skipping" around in order that the next pupil may not study his combination in advance. To maintain the attention of every member of the class throughout the entire exercise, the teacher may permit the pupils to raise their hands promptly when a wrong answer is given, noting those who fail to do so at once, and calling upon one of the negligent ones for the correct answer. As the raising of hands disconcerts some pupils it should not be done in other recitations without formal permission.

Special attention should be devoted to the slower pupils. Before the close of the drill those who have previously made errors may be asked to give the answers to several combinations. If the latter are written on the blackboard in columns and lines, one is asked to take those of the first line, another those of the third line, a third pupil those of the second column, a fourth those of the first column, beginning at the bottom, etc.

In order that all of the members of a class may answer the same set of sight or oral examples, narrow slips of paper are prepared by the pupils, each containing his name, and a row of numbers from 1 to 5 or 10, according to the number of examples to be given.

If the latter consist of combinations already written on the blackboard, the teacher places a pointer on one of them, allows a moment for the answer to be written opposite the number; places a pointer on a second, allows time for the writing of the answer; and so on until all of the 5 or 10 examples are given. When a pupil does not know the result, he is instructed to draw a horizontal line in the place of the answer.

The papers are then corrected by the pupils, each marking his own. The teacher, who has kept a memorandum of the examples, points to the first, and calls upon a pupil for the result. If the answer given is the correct one, the pupils having it on their papers are called upon to stand,

and the number of those standing is written on the blackboard. The pupils having a wrong answer draw a horizontal line through the one written on their papers. This method is followed with all the questions, the teacher determining from the score which combinations need special attention.

The accompanying score by examples shows that a class of twenty-five pupils obtained 221 correct answers out of a possible 250, also that the fifth example proved the most troublesome, and that the sixth and the seventh were answered by every pupil.

Ex.	Correct
1	23
2	19
3	25
4	21
5	18
6	25
7	25
8	24
9	21
10	20
Total..	221

The pupils are then directed to write alongside their names the number of correct answers obtained. The teacher asks those having ten correct answers to stand. Ten (10) is written on the blackboard, and the number standing is placed on the same line. Those having nine correct answers stand, are counted, and the number is written. This is continued until all of the pupils are accounted for, the result being shown in the accompanying score, the total of the second column being 25, which corresponds with the number of pupils. The sum of the products should be 221, as in the first score.

Correct	Pupils	Products
10	15	150
9	3	27
8	1	8
7	0	–
6	6	36
Total ...	25	221

Although this form of recitation requires many words to describe it, it should proceed very rapidly. When the pupils have prepared their papers, the teacher says "Position," which means that the pupils sit erect, their pencils being on their desks. An example is given; after a moment the teacher says "Write," and each pupil takes his pencil, writes the answer at once, or, if he has not obtained the answer, draws a horizontal line opposite the number. The next order, "Position," follows in a moment, the pencils are placed on the desk instantly, and the pupils sit erect. Between the two orders, time sufficient for the pupil to obtain the result from another's paper should not be given. In taking the results, the procedure should be as rapid as possible. The name of the scorer is first announced. "Board, Smith." John Smith quickly goes to the blackboard and writes in a line the numbers from 1 to that of the last question. At the call "Brown," William Brown stands, and reads from his paper his answer to the first question. The teacher says "Stand," and those having the same result stand at their desks. The scorer counts the number standing, and writes it opposite number 1. The teacher says "Sit," and the standing pupils resume their seats. The teacher says "Jones," and Thomas Jones stands. If the answer given by William Brown to the first question is incorrect, Thomas Jones announces his answer. When the teacher says "Stand," it is understood that those who stood when the incorrect answer was given, are debarred from standing again, even though they have the correct answer on their papers and arose through inadvertence. The scorer then erases the number first written and writes the number of those now standing, including himself when he is right, which he indicates by holding up his hand. When the correct answer is finally elicited, the teacher says "Check." Each pupil then takes his pencil, those having the correct answer make a check mark in front of it, while the others draw a line through their incorrect results. At the command "Position" the pencils are replaced on the desks.

A second scorer may be sent to the blackboard to tally the number having 10, 9, 8, etc., correct answers. He begins by placing these numbers on the board. The teacher then says "10 correct," the scorer counts those standing and writes the number. The teacher then says "Sit," and they resume their seats. The next commands are "9," "8," etc., in succession, until each of the twenty-five pupils has stood, which may require the call of all the numbers and also "None." The scorer then adds the column of "pupils," which should give a total of 25. If the scorer is able to multiply by 10, 9, etc., he writes the products and finds their aggregate. Finally the pupils' papers are collected for a subsequent examination by the teacher, to ascertain if a change has been made in any answer, or if a wrong answer has been checked, etc.

It is of great importance that recitations in arithmetic be conducted with a minimum of words by the teacher, the children doing all the work, and doing it quickly and systematically. When children work rapidly, they necessarily give more attention, which develops interest and tends towards accuracy. This is particularly the case in the drill exercises, which are intended to make automatic the pupils' knowledge of the sums, differences, products, and quotients of small numbers.

When pupils have text-books, a set of oral problems may be handled as follows:

At one recitation a pupil reads the first problem aloud and gives his answer; a second reads the next and gives his answer, and so on until the set is finished. If any pupils have not recited, the first of the remaining pupils may begin the set, followed by the others in turn. When the same problems are solved for the second time, the reading aloud may be dispensed with. Whether or not a pupil should be called upon to "analyze" a problem, depends upon its nature. If the set consists of examples involving a single operation, it is generally inadvisable to waste the time.

At a second recitation the answers may be written in the manner suggested on a previous page, the pupils reading silently from their books the problems indicated by the teacher, and writing their answers at command. The same problems or new ones of the same kind should be used in a third recitation, the teacher giving the problem orally, and the pupils writing the answers as before.

When a problem involves several conditions, it is advisable that the slower pupils have sufficient time to solve it. To enable the teacher to determine when all but the very slowest have obtained the result, he requires the pupils to turn over their books as they finish; or, when they are working without books, to sit in "position," with their hands on the desk. The order "Write" is then given, etc.

Except for special reasons, it is advisable to settle each problem after its answer is written, instead of waiting until the set is finished. In this way a pupil having a wrong result may be enabled to ascertain the nature and the cause of the error, and may possibly avoid making a similar mistake in the remaining problems.

The reading aloud of a problem is of great importance to pupils of every grade. Many children do not obtain a proper grasp of the conditions through the eye. As the pupil reads, the teacher notes from his expression whether or not he has grasped the meaning of the words.

Abstract written work should usually be made a class exercise. The example is read by the teacher; when a pupil is done he lays his pencil on the desk, turns his paper over, and sits in "position." When a sufficient time has elapsed, the teacher says "Down," the remaining pupils lay aside their pencils and the others turn their papers face upwards. The answer is called for, the correct pupils stand, and the scorer writes the result on the blackboard. When the recitation is finished, the papers are collected for a subsequent examination by the teacher.

The use of erasers should not be permitted, it being important that the teacher should see all of the figures set down by each pupil, including the mistakes he has made and corrected.

Written problems should generally be solved by the pupils at their seats without help by the teacher, who merely announces the set to be solved. Each pupil should be permitted to work them in any order, selecting first those he understands, and omitting such as are beyond his powers.

As a preparation for the written problems, the teacher should have a recitation in the solution of oral problems similar to those in the written set. He should prepare them beforehand, using the exact phraseology of the written problems, but changing the numbers to bring them within the capacity of the class. To these the answers are written by the pupils in the usual way. At a second recitation, the class should be required to give "approximate" answers to the set of written problems.

The teacher asks, "Who can tell us 'about' the answer to the first problem?" One, who has raised his hand, is called upon; he reads the problem aloud as it is stated in the book, then reads it again, substituting for the given numbers others that he can handle without a pencil, and then announces the result obtained by using the new numbers.

Take, for instance, the following problem, which the pupil reads as it is given in the book:

"If $\frac{7}{8}$ yard of velvet costs $2.45, what will be the cost of $5\frac{7}{8}$ yards?"

His second version of it might be:

"If 1 yard of velvet costs $3, what will 6 yards cost?"

He states that the result in the second version is $18, but that $18 is too great, since 1 yard costs less than $3, and fewer than 6 yards are bought.

Another pupil may be able to substitute $2.80 as the cost of a yard, which would make the price per yard 20 cents less than $3, making the cost of 6 yards $1.20 less than $18, etc.

Another might employ the ratio method, as is shown in the following:

"If 7 eighths of a yard of cloth costs 2\frac{1}{2}$, what will be the cost of 49 eighths of a yard?"

The attempts to obtain close approximations will cause the pupils to read the problem intelligently. The important thing in this procedure is for the average pupil to realize at once that a whole yard will cost more than $2.45, the customary impulse of the heedless pupil being to multiply by a fraction instead of dividing. Even the pupils that do not volunteer a solution are learning more through the attempts of their classmates than they would from the teacher's explanation. These preliminary recitations are intended to teach the pupil to change the numbers of a written problem to small ones from which to determine the required operations, and then to employ numbers nearer to the given ones, from which he may "approximate" the result, and thereby check his work.

Before a new set of problems is assigned, there should be a recitation in which the previous set is solved upon the blackboard by such pupils as have not obtained the correct results. The first reads an example from his book and works it. If he completes it without a mistake, the teacher inquires if any one can solve the same problem in a shorter or a better way, or even in a different way. The teacher should then inquire if any pupil is still unable to understand any one of the solutions upon the board, and give him an opportunity to state his difficulties.

The requirement in some examination papers that all work should be shown and that every step should be labeled is unwise. No pupil should be permitted by a teacher to write unnecessary words or figures. The teacher should so arrange his pupils that they are not unfairly tempted to look at a neighbor's paper. For this reason it is advisable that pupils doing the same work should sit one behind the other, when possible, instead of side by side. When a class is divided into two sections, one section, seated as above, can work desk problems, while the other is doing abstract exercises under the direction of the teacher. Both sections can be brought together for such recitations as are best treated as class exercises. A pupil is seriously injured by being permitted to help himself to an improper look at another paper, and the teacher is blamable who does not do all in his power to make it impossible, instead of satisfying his conscience by simply forbidding it.

The division of a class into sections is necessary for various reasons. The slower pupils should form a section for many exercises, in order not to delay the quicker ones. It is of supreme importance that every pupil learn how to perform the operations belonging to his grade, and to do so with accuracy. Rapid work is useful, but not essential. The slow pupil should be required to obtain correct answers to examples in addition, subtraction, etc., even if it is necessary for him to make marks on his slate, count on his fingers, consult the multiplication tables, or the like, all of which the quicker pupils find unnecessary through the various drills, the short time allotted for each example, etc.

Promotion should be open to the slow but accurate pupil just as freely as to the quick one, provided the former is mature enough to take up advanced work, even if he is too sluggish mentally to combine numbers as rapidly as many of his more gifted classmates. The amount of number work required in later life by the average man is limited, and to deprive a pupil of an opportunity to pursue other studies because of a failure to reach an artificial standard in mathematics is a serious educational sin.

Real life demands a rating of 100% in nearly all calculations, but it generally gives the calculator all the time he requires. The teacher must, therefore, secure this to the greatest extent possible, rapidity being exacted only in so far as it tends to make work more accurate by calling for greater attention. For this reason pupils should not be required to dissipate their attention by writing unnecessary numbers or words, but each, according to his capacity, should be permitted and encouraged to set down everything the teacher finds it necessary for him to write in order to obtain a correct answer. Short methods are merely for those who will be benefited by their employment.

As an exercise in composition, the occasional written analysis of a given problem is a useful exercise in requiring the pupil to express himself clearly and accurately. To limit, however, the number of problems solved by a pupil by requiring him to write out each step and to take superfluous steps at the whim of an examiner, is to do the pupil an injustice. It may be that some pupils make blunders in their problem work by their neglect to do writing enough, and the teacher should call their attention to the means by which these mistakes may be avoided, but every pupil should not accommodate his work to the abilities of his duller mates. The blackboard recitation shows what aids the various pupils need.

Blackboards are used in different ways, according to the prevailing conditions. The modern school is so abundantly supplied that there is room for a whole class to work at the boards simultaneously. This gives the opportunity for the teacher to see how each does his work. When all are at the boards together, a different exercise may be assigned to each, who takes the assigned example from his book. In this case the book should not contain the answers.

It is important for pupils to be shown the various items which tend to retard their calculations, and which, for this reason, should be omitted. While some of these things are disclosed by an examination of their papers, others escape the teacher's attention until they are observed during the blackboard work. Neat arrangement of the work does not require the use of the ruler in drawing lines, and the like.

The blackboards may sometimes be used by the slower section while members of the other section are working examples at their desks. Each pupil in the former section is permitted to continue at his example until he obtains the correct result, while those of the other section cease work at command.

In order that the time of the pupils may be fully occupied, the teacher should not withdraw himself from his duty of instruction to endeavor to make any memoranda of the number of examples correctly worked by his pupils. If, for any reason, he desires this information, he can obtain it after hours from an examination of the collected papers. For most purposes he needs to know only in a general way what effort a pupil makes and what success he attains. A letter denoting each can be placed against a pupil's name at the end of the month or week. The relative per cents of the work of all of the pupils of a class are misleading and, therefore, worthless.

CHAPTER V

GENERAL SUGGESTIONS

NOTHING is more characteristic of the successful teacher than his ability to keep his pupils profitably employed at all times. This he is able to do by his habit of determining in advance the procedure to be followed each day. If the pupils use a text-book, he looks over the examples, and notes those that should be omitted or postponed, and provides such additional ones as may be needed, being careful to adapt them to the capacity of his pupils. He outlines the successive points he desires to make in each new lesson, leaving nothing to the inspiration of the moment, which his experience has shown him to be unsafe.

The required sight exercises should be written on the blackboard before the arrival of the class, and covered with a map or chart until the time for their use is reached. Combinations may be written with charcoal on a large sheet of paper, to be placed before the class when needed. Cards containing the new combinations are added to the existing set for use in later drills.

The illustrative material required for a new lesson should be at hand, and the teacher should from time to time make additions to his stock.

When the school does not own a set of measures, they should be borrowed for the lesson in which the pupils learn the relation between the pint and the quart, etc., by experiment.

When objects are needed to clear up any point, the teacher should be sure that they are at hand. In many cases the pupils, the desks, the panes in the window, etc., are more interesting to young children than cubes and splints, and the teacher should determine before the lesson what objects he will employ.

Problems based upon the activities of the children, the industries of the vicinity, etc., should be noted in a memorandum book kept by the teacher, to supplement those of the text-book. Bills for purchases actually made are useful to show the exact form employed by business men. Material should be supplied by members of the class as far as possible, the search therefor developing in the pupils an interest in a lesson illustrated through their coöperation.

The teacher should bear in mind that a text-book contains matter which it is not expected that the pupil should use at the point at which it is found. The preliminary definitions, for instance, which are rarely of value at any time, are particularly worthless as an introduction to a new topic. Under no circumstances should the teacher waste the time of young pupils by requiring them to memorize the crude forms prepared for beginners, and which later on would have to be replaced by more accurate ones. The child has no more need for the definition of addition, division, fraction, etc., than he has for one of mother, chair, school, field, or horse. The German text-books in arithmetic contain nothing but carefully graded examples and problems, the number and variety of which are made possible by the omission of all definitions and rules. The teacher should understand also that the arrangement of the details of a given topic as found in the text-book does not necessarily show the order in which these should be studied. The arrangement that follows the logical development of the subject matter may be convenient for a pupil who uses his text-book as a work of reference in reviewing his previous lessons, while not the one to be followed by a teacher.

Although pupils should not memorize the definitions of the text-book, they should be required, at the proper time, to employ the necessary technical terms correctly. They learn to do this in the way they learn to talk, by repeating the words used by the teacher. While the latter should not unnecessarily introduce these terms, he should not refrain from using them when occasion requires, and he should not feel obliged to employ a substitute that is manufactured for infant consumption.

The early work of adding $\frac{1}{2} + \frac{1}{2}$ does not require the use of the word "fraction." The addition of $\frac{1}{2} + \frac{1}{4}$ can be performed by a pupil before he hears of "denominator," "common denominator," or "least common denominator." When it becomes necessary in a later grade for the teacher to speak of a "denominator," the pupil learns to recognize it as the number below the line, and to employ the term himself. He is not benefited at this time by an etymological analysis of the word, which belongs to a still later stage. The good teacher keeps constantly in mind that unnecessary diversions from the work in hand tend to lessen the pupils' interest in the subject and to waste their time.

A teacher should have a reason for everything he does. He should not assume, for instance, that drills in counting backwards are as useful as those in counting forwards. The latter are essential, while the former are subordinate; and when the addition tables are known, subtraction practically takes care of itself. This does not mean that there should be no drills in finding the difference between two numbers, but it does mean that these should not be had at the expense of the more important ones. The same is true as regards multiplication and division drills. When the pupil knows thoroughly the multiplication combinations, he soon becomes able to obtain quotients. Division drills should be used, but not to the same extent as the others.

The requirement that answers should be given in complete sentences is a bad one, so far as arithmetic is concerned. Anything that requires a pupil, when asked the sum of 7 + 4, to say "7 and 4 are 11," tends to deprive him of the benefits aimed at in the drills. It gives him an excuse to repeat the numbers slowly, thereby obtaining time to count, etc. The first thing a teacher should insist upon in an oral exercise is a prompt answer, expressed in the fewest possible words, preferably merely the number with the accompanying "houses," "acres," and the like.

When called upon to give his analysis, the pupil may repeat the whole question, again give the result, with the steps taken, and the reasons; but this is done after the immediate response has been secured.

The analysis of an oral problem may be required for several reasons. One may be the desire of the teacher to be assured that the pupil has obtained the result by himself; but, as has been said before, it should be made almost impossible for a child to avail himself of the assistance afforded by a glimpse of another's work. Oral analysis by a pupil is useful chiefly as an aid in teaching, and it should be employed solely for this purpose. In the early years, when the required operation is obvious to every child, and the errors are due to faulty acquaintance with the tables, nothing is gained by having pupils analyze each example. Once in a while a problem may be analyzed in order to accustom pupils to this phase of the recitation as a preliminary to its regular employment later on, but it should be used sparingly and infrequently.

In later grades the teacher may call upon a pupil who has the correct result to state how he obtained it. After repeating the problem and giving the answer, he states in his own language the steps he took. A prescribed form is worthless, with its superfluous words that obscure the explanation. An analysis may be advisable when a class is given for the first time a problem whose solution requires two operations and when several pupils have obtained wrong results. For instance, take the example,

When 2 oranges cost 8 cents, what is the cost of 7 oranges?

The type frequently recommended is: "When 2 oranges cost 8 cents, 1 orange will cost $\frac{1}{2}$ of 8 cents, which is 4 cents, and 7 oranges will cost 7 times 4 cents, which is 28 cents."

Some pedagogues insist that this analysis is defective, inasmuch as it omits important links in the chain of reasoning. They advise the following:

"When 2 oranges cost 8 cents, 1 orange, which is $\frac{1}{2}$ of 2 oranges, will cost $\frac{1}{2}$ of 8 cents, which is 4 cents, and 7 oranges, which are 7 times 1 orange, will cost 7 times 4 cents, which is 28 cents."

The teacher that is better acquainted with the child prefers the following type:

"When 2 oranges cost 8 cents, 1 orange costs 4 cents, and 7 oranges cost 28 cents."

This wording indicates merely the successive steps; namely, that from the given cost of 2 oranges that of 1 orange is found, from which the cost of 7 is obtained. Teachers using this form consider it unnecessary to analyze a one-step problem, since pupils obtain readily the value of 1 when that of 2 is given, and the value of 7 when they know the value of 1.

In order to ascertain that the analysis has been of assistance to a pupil who has not obtained the answer, the latter is called upon to restate the problem, give the answer, and analyze it. If he succeeds, the teacher may give him a problem of the same kind, but containing other numbers;

assuming, if he gives the correct answer, that he has understood the explanation. When the problem can be solved in several ways, it is advisable to ask a pupil who has obtained the answer in a different way to state his method, thus giving the backward pupil a chance, if several methods are presented, to hear one that he may be able to apply to other problems. A pupil's method should be stated in his own way in the first instance, to be modified by him, through the teacher's questions, if it is defective or too long drawn out.

The form of the first of the three types given above is defective in the employment of $\frac{1}{2}$, which is not one of the numbers found in the problem, and it is advisable to limit the young pupil to the numbers given. Its use in the second is unnecessary and is, therefore, inadvisable; since the ratio method should be used in lower grades to explain only such oral examples as the following:

Find the cost of 12 pears at the rate of 3 for 5 cents.

In analyzing this, the pupil may say that 12 pears should cost 4 times as much as 3 pears, or 20 cents. The extremist insists upon too much verbiage in the wording of some problems. He insists upon the introduction of the word "average" in an example asking how many miles an hour does a boy travel if he covers 8 miles in 4 hours. While the attention of older pupils may be directed to the fact that a boy able to walk 3 miles in an hour cannot walk 24 miles in 8 hours, it is unnecessary with young children to obscure a simple example by the introduction of words to them unnecessary.

Another form of pedantry consists in requiring that an analysis of a multiplication example should clearly point out the abstract multiplier. Thus, the cost of 21 pounds of 4-cent flour should be analyzed in their opinion as "21 times 4 cents," and not that "the number of cents in the cost is 4 times 21." Those who permit the use of 4 as a multiplier teach this form: "At 1 cent per pound, 21 pounds would cost 21 cents, at 4 cents per pound the cost is 4 times 21 cents, or 84 cents," but not so briefly expressed.

When the pupils study algebra they are not bothered with concrete numbers, the writing of the denominations, and the like; they use abstract numbers throughout, rewriting the result with the proper word attached; cows, pounds, men, etc. In the same way, when they come to the study of geometry, they learn that the area of a rectangle is equal to the product of the base and the altitude, with no warning against the so-called error involved in multiplying inches by inches.

Arithmetic exercises should not be assigned to be done at home. Pupils may use their books to make any necessary preparation, but written examples should be worked in the class under the eye of the teacher.

To secure attention, the teacher should announce the numbers in an example but once. It may be that the repetition of a strange written problem might be advisable, the first statement of the teacher being made to give the pupils an opportunity to grasp the conditions without paying particular attention to the numbers, and the second being intended to supply the numbers, which they should not be permitted to write during the first reading.

The pupils of the lower grades should be required to solve comparatively few written problems, and a great many oral ones. The latter need not necessarily be confined to the topics used in the regular written exercises of the grade. A pupil may be asked how many quarters there are in a half pie some time before he reaches fractions. and a teacher should not feel that he does wrong in accepting the correct answer from a pupil without taking advantage of the opportunity to talk of reductions, etc. A pedagogue who finds in an article on methods a lengthy statement of the way the addition of two fractions should be taught, and declines to permit a pupil to give the correct answer to $\frac{1}{2} + \frac{1}{4}$ without the development exercises suggested by the writer of the article, is not doing the right thing by his pupils.

SECOND PART — WORK OF FIRST FOUR YEARS

CHAPTER I

NUMERATION AND NOTATION

During the early number lessons outlined in the next chapter, the reading and the writing of numbers of one figure are taught incidentally. The same course is followed in teaching the numbers from 10 to 13, inclusive. There is no need at this stage to allude to the place value of the figures. The fact that 13 is equal to 10 and 3 is a part of the addition work, and the pupils know sufficiently well for their purpose that the figure 1 before 0, 1, 2, etc., stands for 10, and that the left-hand figure means 0, 1, 2, etc., ones, as the case may be. The mistake of the young teacher consists in requiring pupils to formulate statements of general truths at too early an age.

When the pupils are reasonably familiar with numbers from 10 to 13, they learn the others in a few minutes. The teacher writes, for instance, 16 on the blackboard, and calls upon a pupil to read it. She sends another to the board to write eighteen. When a child is asked to read 15 for the first time, he may be tempted to say *fiveteen*, but he quickly corrects himself.

The excellence of a teacher is shown as much by what he omits to say as by what he expresses. Children develop by what they do for themselves, and the average child resents being told unnecessary things, as much as does his elder the unnecessary elucidation of the evident point of a joke.

The teaching of the numeration and the notation of the remaining numbers of two figures is a simple task. A pupil is asked to count by 10's, to 90, something each can do; another counts from 20 to 29; a third from 50 to 59, etc. The teacher then writes 20 on the blackboard, and asks who can read it.

Successive pupils are sent to the board to write sixty, eighty, seventy, ninety; thirty, forty, and fifty being left for the last. Then, other pupils in turn are called upon to write eighty-three, twenty-five, forty-four, etc., etc.

The numbers written on the board are next used as a drill exercise in reading numbers. Examples in addition give practice in writing these numbers as well as in writing those of the succeeding grades.

When numbers of three figures are prescribed by the course of study for a grade, the teacher develops the need of a new name by calling upon a pupil to count by ones from 90. If he hesitates after 99, volunteers are asked for to give the next number. When this is obtained a pupil is asked to count by hundreds to 900; another, by tens, from 100 to 190; a third, by tens, from 500 to 590; a fourth, by ones, from 320 to 329; etc., etc.

The next step (in a city school) may consist in asking what pupils have a house number in the hundreds, for instance. Different pupils announce their numbers, and are sent to the board to write them, which the average child of this grade can do correctly. A pupil may be asked how many figures always follow the figure that tells the hundreds. Numbers containing ciphers are next written on the board by the pupils, the teacher first dictating numbers having a terminal cipher such as: 150, 390, 570, 760, etc. Later come such numbers as 203, 401, 608, 804, etc., as well as the even hundreds: 100, 200, 300, etc. The numbers on the board supply reading drills, and the examples in addition, etc., the written drills.

In reading numbers of three figures, the word *and* should not be employed. Children should be required to say one hundred five, three hundred seventy-seven, etc., without using the unnecessary conjunction.

Numbers of four figures are taught in the same general way pursued in teaching numbers of three figures. The need for the word *thousand* is developed by calling upon a pupil to count by ones from 990. If he cannot continue after 999, and no one can give the name of the next number, the teacher states that it is one thousand. A pupil counts by thousands, from 1000 to 9000, another by hundreds, from 100 to 1000, a third by hundreds, from 1000 to 2000, the use of eleven hundred, twelve hundred, etc., not yet being permitted.

After many such counting exercises, the teacher asks what pupil can write two thousand three hundred forty-five, for instance. When this is written correctly, the teacher directs that a comma be inserted after the 2. The members of the class recognize the assistance supplied by the comma, and different pupils readily write other such numbers consisting of significant figures exclusively. The fact that three digits always follow the comma is brought out, and numbers containing ciphers are written on the blackboard at the teacher's dictation, the first numbers being those containing a terminal cipher, then those having a cipher in the tens' place, then those having a cipher in the hundreds' place, then those having two terminal ciphers, lastly those having three terminal ciphers.

Although it takes many words to outline the foregoing lesson, the time spent in teaching it need not be very long if the teacher keeps things moving rapidly and refrains from unnecessary talk. He should be firm enough to resist every temptation to descant upon the beauties of the Arabic system, the place value of the figures, and the like, no matter how much he is advised to do so by people who do not know the child mind. Only the directions of his superiors should induce him to waste the time of babes in trying to interest them in general truths.

The excellence of the Arabic system is due to the fact that it makes it unnecessary for the business man to think of ones, tens, hundreds, etc., in any of his calculations, and why should the pedagogue introduce artificial difficulties into the work of a child? Later in the course, when numbers need division into periods before they are easily read, it is advisable to speak of ones, tens, hundreds, etc.; but unnecessary details should be omitted until a need for their introduction arises.

When, however, the teacher finds it necessary to call attention frequently to a digit in the tens' place without giving the name of the figure, he need not hesitate to say "tens' figure," without explanation if the pupils recognize it under this title. The expression "ones' figure" is not so necessary at first, as it may be called the right-hand figure. The word *one* is generally preferred to *unit* in this connection, since a second meaning for the latter comes into use later on, and *ones* corresponds with *tens*, *hundreds*, etc.

Numbers of five and of six figures need no special treatment after those of four figures are readily written and read. As the latter are generally continued throughout the work of a half year, or longer before the larger ones are introduced, these can be written at once by the average boy or girl. A pupil that can read and write four thousand can read and write forty-four thousand and four hundred forty-four thousand. Such numbers as the latter may be dictated to pupils in addition and other examples in the proper grades, without any preliminary exercises.

The reading and the writing of numbers containing dollars and cents, which constitute the introduction to formal notation and numeration of decimals, are generally taught in the third year. In some of the earlier work the pupils have become acquainted with the dollar mark, and have learned that it is usually prefixed to a number instead of following it, as in the case of other abbreviations; such as, those for pounds, gallons, cents, etc. When the time comes to combine dollars and cents into a compound number, the teacher asks, "What girl can write on the blackboard four dollars and fifty cents?" If it is not written properly, the teacher calls for corrections. When it appears in the form, $4.50, the teacher elicits the information that the abbreviation for cents is replaced by the period, the name of which he gives as the *decimal point.* In the addition examples of this grade, the pupil learns that 90 cents is written $.90, when it is used in connection with other addends composed of dollars and cents. He finds also that it should be so written even if the addends consist exclusively of cents, when the sums exceed 100 cents. From this time on, the decimal form of indicating cents may be regularly employed. There is as yet no particular necessity for discussing dimes and cents, tenths and hundredths, unless the course of study prescribes two-place decimals at this stage.

The chapter headings in the pupils' readers and the numbers on the face of the clock give the pupils an acquaintance with some of the smaller Roman numbers. The fact that I signifies one, V five, and X ten, is brought out by questions, also the meaning of II, III, IIII, VI, VII, etc., to XII. As Roman numbers are not employed in calculations, the pupils of this grade should not be drilled in writing them, although they might well use them to indicate on their papers the number of a problem.

In another grade Roman numbers may be carried, perhaps to twenty. At this point it may be necessary to show that four is written IV, except on the clock-face. The fact that numbers from eleven to nineteen, inclusive, are expressed by placing after X the letters indicating those from one to nine, is then told, if necessary, and successive pupils write these numbers on the blackboard. Others are asked to read numbers written by the teacher. Later the reading of XX and XXX may be called for, also the reading and writing of other numbers to thirty-nine. The representation of fifty by L must generally be stated by the teacher, also of forty by XL. It is not yet necessary to formulate the rule that I preceding V or X is subtracted, or that the same is done when X precedes L or C; it is sufficient for the present to have the pupils recognize IV, IX, XL, and XC, leaving the rule for a later stage. But little time should be given to this topic.

CHAPTER II

ADDITION

The average child, when he enters upon the work of the lowest grade in which number is taught, has learned to count to some extent. In the first lesson each pupil is called upon in turn to say 1, 2, 3, etc., as far as he can, being stopped, however, at 9. Owing to their varying ability in this respect, it may be necessary to form two divisions. The next step is to ascertain if these numbers mean anything to the child; if 2, 3, for instance, are any more than mere names.

The earliest lessons should require some physical activity on the part of the child, such as walking, making marks on the blackboard, giving taps on the desk, and the like. A pupil should be asked, for instance, to walk three steps, another to make three chalk marks on the board, a third to give the desk three taps, a fourth to touch three classmates, a fifth to pick up three books, etc. The counting of the number of panes in the window, of legs to a chair, of objects placed on a table by the teacher, constitutes another type of exercises, to be employed at the proper time.

Regular work in addition is begun by requiring pupils to give answers to oral problems, each providing for the increase of a given number by 1. The teacher commences with a number known to every member of the class or section, say 3. When he finds that each pupil knows three steps, three marks, three objects, etc., he directs a pupil to walk three steps and then to walk one more, and asks how many steps have been taken. If this one does not know the answer, it is obtained from another pupil.

Then come such oral problems as the following:

(*a*) James has three marbles; how many will he have when he gets one more?

(*b*) Mary's sister is three years old; how old will she be in one year more?

A large number of these problems with "4" for an answer should be given by the teacher. The answers should not be in complete sentences, but should give the denomination: marbles, years old, etc.

In order that pupils may not always be able to give the answer as "four" without carefully listening to the problems, the numbers should be so varied as to call for answers of 2 and 3 also, the conditions in every case, however, requiring only the addition of 1 to another number.

To enable pupils to add numbers at sight, the writing of the numbers 1, 2, and 3 may now be taught, and these combinations written on the blackboard and also on cards to be used as recommended in a preceding chapter. As a preliminary to these sight exercises and accompanying them, oral answers are required to such questions as:

$$\begin{array}{ccc}1 & 1 & 1\\ \underline{1} & \underline{2} & \underline{3}\end{array}$$

(*a*) One and one are how many?
(*b*) Two and one are how many?
(*c*) Three and one are how many?

to make the children acquainted with abstract numbers.

The suggested procedure takes this order:

1. To perform three acts, and then to do 1 more.
2. To count 1, 2, 3, 4.
3. To solve problems requiring the addition of 1 to 1, to 2, and to 3, respectively.

4. To give answers to abstract questions asking the sum of 1 and 1, 2 and 1, 3 and 1.

5. To give answers at sight to the same combinations.

The numbers to be combined should be written as indicated, to aid pupils in written addition, in which the second addend is written under the first. The larger number is placed underneath, to accustom pupils to commence to add from the bottom upwards.

The next step is to give such oral problems as the following, the first number in each being 1:

(*a*) William has one cent; how many will he have after he gets two more?

(*b*) Jane's brother is one year old; how old will he be three years from now?

When the answers to a number of these have been given, the abstract questions follow:

(*a*) One and two are how many?
(*b*) One and three are how many?

and these combinations are added to those previously taught. The fact that 3 added to 1 gives the same result as 1 added to 3 should not be alluded to at present; the pupils will obtain this knowledge for themselves.

$$\begin{array}{cc} 2 & 3 \\ \underline{1} & \underline{1} \end{array}$$

Each successive number to 8 is increased by 1 in subsequent lessons, following the plan given above; and 1 is increased by 4, by 5, by 6, by 7, and by 8. The combinations at this time will include the following:

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} \end{array}$$

together with these:

$$\begin{array}{ccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \end{array}$$

The next stage consists in teaching the increase of 2, 3, 4, 5, 6, and 7 by 2, each successive combination being given the requisite number of lessons. In the first, which may require a pupil to walk 2 steps, then 2 more, and to give the result, he will actually say 3 to himself before giving the answer 4; the tendency to think of the intervening number will, however, be gradually eliminated by requiring the answers to be given more and more rapidly.

The oral problems and exercises follow in turn, then the following combinations for drill:

$$\begin{array}{cccccc} 2 & 2 & 2 & 2 & 2 & 2 \\ \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} \end{array}$$

The teaching of the increase of 2 by 3, by 4, by 5, by 6, and by 7 is commenced by oral problems. These lessons do not require so much time as the ones immediately preceding, since the pupils realize the fact that they already know the same combinations in another form. Oral abstract examples, etc., come next, to be succeeded by the following drills:

$$\begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \\ \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} \end{array}$$

As pupils can readily count to 9, they may be given problems requiring the addition of 3 to 3, to 4, to 5, and to 6, the answers to which they may be permitted to obtain by using splints, cubes, or other suitable objects, with which they have been supplied. The reverse problems may also be solved in the same way; namely, the increase of 3, by 4, by 5, and by 6. The use of objects should not be permitted in the solution of problems involving the earlier combinations. Pupils too sluggish mentally to have mastered them by this time, will probably use their fingers. The teacher should not hope that every pupil will attain the desired expertness, and the dull pupil may be allowed a trifle longer time for his answer without any allusion to his lack of speed. Oral problems involving the foregoing combinations are then given in the way suggested for the preceding ones, but the answers need not be required too rapidly at first. The drills will fix the necessary facts, and the time permitted for answers is lessened from week to week.

The foregoing include these combinations:

$$\begin{array}{cccccccc} 3 & 3 & 3 & 3 & & 4 & 5 & 6 \\ \underline{3} & \underline{4} & \underline{5} & \underline{6} & \text{and} & \underline{3} & \underline{3} & \underline{3} \end{array}$$

The following combinations:

$$\begin{array}{cccc} 4 & 4 & & 5 \\ \underline{4} & \underline{5} & \text{and} & \underline{4} \end{array}$$

are all that remain to complete the series of two numbers whose sum does not exceed 9. Examples that involve such combinations as (*a*) 0 + 5, (*b*) 3 + 0, (*c*) 0 + 9, should not be given. The cipher is best introduced with two other addends, as is shown on the next page.

The preceding oral work is deemed sufficient for the first half of the first grade, the remaining combinations of two numbers of one figure each being left for the second half. When the beginning of arithmetic is postponed until a later grade, all of the combinations to 9 + 9 may be learned in less than a year.

Since the first examples in written addition are limited to column totals no greater than 9, some teachers take it up during the first term, while others postpone it for a half year.

As a preliminary to written addition, the pupils may be permitted to write answers on the blackboard to exercises such as the following:

$$\begin{array}{ccccccccc} 1 & 2 & 5 & 4 & 2 & 3 & 3 & 4 & 6 \\ 1 & 3 & 2 & 2 & 2 & 1 & 3 & 5 & 0 \\ \underline{7} & \underline{1} & \underline{1} & \underline{3} & \underline{2} & \underline{4} & \underline{3} & \underline{0} & \underline{2} \end{array}$$

the cipher being now used in some of them; but oral problems should not, as a rule, contain more than two numbers at this time.

When the pupils have learned to read and to write numbers of two figures, an example may be worked at the blackboard by a pupil, who is directed by the teacher to write 32, and under it to write 32, and under the second addend to write 32. He is then told to add the 2's and to write the sum in the same column. He is next told to add the 3's and to write the sum of this column. Examples similar to the following are written and worked by successive pupils:

$$\begin{array}{rl} 32 & \\ 32 & \\ \underline{32} & \\ & \textit{Ans.} \end{array}$$

$$\begin{array}{rrrrrr} 23 & 36 & 4 & 3 & 15 & 30 \\ 52 & 2 & 70 & 3 & 21 & 20 \\ \underline{4} & \underline{60} & \underline{14} & \underline{82} & \underline{3} & \underline{30} \end{array}$$

The reason for placing the addend 4 in the ones' column in the first example should not be explained by the teacher. If the pupil makes a mistake in this respect, the teacher may ask, "Where is the 3 in 23 written? Where is the 2 in 52 written? Where should 4 be written?" An example already worked may be left on the board as a guide for the pupil in working the next.

The successive steps in adding may be shown by having the pupil announce each step, but he should be required to omit unnecessary numbers. In the first example he should say only, "4, 6; 6, 9"; and the final total, "Answer, 96." In the second, "6, 9; 7; answer, 79." In the third, "8, 9; answer, 98."

Examples at the end of the term may contain four addends, provided always that the sum of each column does not exceed 9, and that no number contains more than two figures.

There are in all forty-five different combinations of two numbers of one figure each, of which twenty have already been taught. The remaining twenty-five include the addition of 1 to 9; of 2 to 8 and to 9; of 3 to 7, to 8, and to 9; of 4 to 6, to 7, to 8, and to 9; of 5 to 5, to 6, to 7, to 8, and to 9; of 6 to 6, to 7, to 8, and to 9; of 7 to 7, to 8, and to 9; of 8 to 8 and to 9; and of 9 to 9. The reverse combinations should also be taught. The method of teaching each should be the same as that pursued in the other combinations; the time spent on each should, however, be gradually reduced.

Supplementary to the drills in the foregoing combinations; and to give variety to this kind of work, counting drills are employed, which are useful in developing the ability to add rapidly. They consist in counting by 2's, 3's, 4's, etc., to 9's, each pupil carrying one set to a given point, and the next taking a different set, or a pupil announcing one result and the next pupil another. At this stage the highest total may be made 19. In counting by 2's, the first set is 2, 4, 6, etc., to 18; the next, 1, 3, 5, 7, etc., to 19. In counting by 3's, the first set is 3, 6, 9, etc., to 18; the next, 1, 4, 7, etc., to 19; the next, 2, 5, 8, etc., to 17. In counting by 4's, there will be four sets, one beginning with 0, the next with 1, the next with 2, and the last with 3.

In these drills the pupil gives merely the totals, not being permitted to name the two addends. There is, however, an occasional pupil who appears to need the aid of the voice and the ear, as well as that of the eye. Such a pupil should be permitted to say aloud the numbers before giving the total, but he should not do so during the regular recitations. The intelligent teacher will take care that a boy or a girl needing more time and attention than the other pupils receives both as far as conditions permit.

If written addition without carrying is taught in connection with oral work to 9, addition "with carrying" may accompany the combinations to 18, each being made to aid the other. At first, the addends will be limited to two, the sum not being greater than 99, and the total of the ones' columns not exceeding 18. Some teachers prefer to use at the start one addend smaller than 10. While it is inadvisable to give many examples as written work when the answers can readily be determined by inspection, the beginning might be made by having a pupil write 2 on the blackboard and 16 underneath. To prepare the pupils for the new operation with its accompanying sign, the plus sign (+) should appear in the new combinations, with the statement: "This is called plus; it shows that we should add."

$$\begin{array}{r} 2 \\ +16 \\ \hline \end{array}$$

The pupil is told to place the plus sign before 16 and to read the example. This he does, as "16 plus 2," or "2 plus 16." He is then asked to give the answer, and he should be able to say that it is 18. He is now required to show how he would get the answer by adding the two numbers as he did in the other examples. This he proceeds to do by writing 8 as the sum of the ones' column, and 1 in the tens' column. If he writes 1 first, he should be asked if he is commencing in the right place.

The same pupil, or another, may be asked to combine 2 and 17 in the same way; then 2 and 18. The average boy or girl will write 10 as the sum of 8 and 2, and proceed to place another 1 in front of the 10. The teacher intervenes to say that the 1 in 10 is not written, but that it is "carried" to the other 1, making 2, which is written in the second column. As this result, 20, coincides with the pupil's mental

$$\begin{array}{r} 2 \\ +18 \\ \hline 10 \\ 1 \\ \hline \end{array}$$

addition of 18 and 2, he accepts this method of obtaining it.

Successive pupils are sent to add 18 and 3, 19 and 2, 4 and 17, 5 and 15, 19 and 3, etc., in order that the method may become fixed. The sum of 25 and 25 will furnish an example of two numbers of two figures each, which may be followed by many others, worked upon the blackboard until the pupils become familiar with "carrying." The ones' figures of the numbers to be added should be those of the combinations already learned or being taught at the time. The total of the tens' figures will be 9 or less for the present.

When the notation and the numeration of three figures are learned, the examples may consist of addends of two figures each, whose sum is 100 or more; such as 25 and 75, for instance, from which the pupils learn to write the entire sum of the last column, including the 1 carried. Some examples should be included in which the sum of the ones' column is less than 10, to teach them that carrying does not always occur. These may comprise such as,

$$\begin{array}{r} 25 \\ +75 \\ \hline \end{array}$$

$$\begin{array}{r} 84 \\ +\ 35 \\ \hline \end{array} \qquad \begin{array}{r} 63 \\ +\ 52 \\ \hline \end{array} \qquad \begin{array}{r} 48 \\ +\ 60 \\ \hline \end{array} \qquad \begin{array}{r} 29 \\ +\ 40 \\ \hline \end{array} \qquad \begin{array}{r} 72 \\ +\ 15 \\ \hline \end{array}$$

There may be also some examples containing three or more addends, some of them containing three figures and others only two. The teacher's preparation provides examples, progressive in difficulty for each recitation.

$$\begin{array}{r} 123 \\ 25 \\ 260 \\ \hline \end{array}$$

The following outline shows a tentative grading of work in written addition for the first four years:

Grade	Term	Number of Addends	Column Total	Sums not Exceeding
1st	1st	2	9	9
	2d	3	19	99
2d	1st	4	29	999
	2d	5	39	999
3d	1st	6	49	9999
	2d	7	59	9999
4th	1st	8	69	99999
	2d	9	79	99999

The foregoing table assumes that number work begins with the child's first year in school, and gives the maximum that should be done in four years. When the study of arithmetic is postponed for a term or two, the necessary modifications should be made.

The outline provides such limitation in the number of addends, in the total of each column, and in the number of columns, as should be observed in order to keep the work well within the powers of the average child. There is no good reason for making the work of any term more difficult than is given in the foregoing maximum requirements. These provide for the largest amount of sustained attention that should be expected from the normal child of each grade. A third addend in the sight work at the end of the first term is allowable, since each example contains but one column.

The writing of carrying figures or of helps of any kind should not be tolerated. It is unnecessary for the average pupil and tends to poor work. When the columns are long, business men employ carrying figures, but for different reasons; and in the upper grades their proper use is desirable. Marks on the paper, counting on the fingers, and the like should be discouraged.

Results in addition are tested by adding the columns downwards if the original result is obtained by adding the columns upwards. The first answer should be covered by a piece of paper until the second is obtained. If the two agree, they are in all probability correct.

At some stage in the work of the first year are introduced oral combinations of two numbers whose sum is less than 20, and one of which exceeds 9. Although easier than many of the special combinations of the grade, such as 8 + 9, etc., these should not be permitted to interfere with the mastery of the others, which is essential to all future progress. A person knowing the sum of 6 and 3 can ascertain that of 26 and 3, etc., by written addition, in case he cannot do so without a pencil. The sum of 8 + 9 should, however, flash into the mind the moment the numbers are seen or heard.

These new combinations are first used in oral problems. As sight exercises they should be written in the equation form, (*a*) 12 + 7, (*b*) 8 + 11, (*c*) 13 + 5, (*d*) 3 + 14, in order that pupils may not be tempted to begin with the ones' figures. If, however, the teacher feels that the pupils should continue to see the addends in the form employed in written addition, the following arrangement is preferable when the pupils begin to add from the bottom; but subsequent combinations should be written on a line to familiarize pupils with the sign, and as an introduction to later examples in addition in which one addend is not written under the other.

$$\begin{array}{ccc} 7 & 8 & 5 \\ \underline{12} & \underline{11} & \underline{13} \end{array}$$

The oral exercises of the second year generally extend these last to include the addition of a number of two figures to a number of one figure, the sum not to exceed 99. One feature comprises the so-called "decimation drills," the first of which do not involve carrying; such as,

(*a*) $5 + 3$, $15 + 3$, $25 + 3$, $35 + 3$, $45 + 3$, etc.
(*b*) $4 + 5$, $14 + 5$, $24 + 5$, $34 + 5$, $44 + 5$, etc.
etc. etc. etc. etc. etc.

Then come the following:

(*c*) $5 + 6$, $15 + 6$, $25 + 6$, $35 + 6$, $45 + 6$, etc.
(*d*) $8 + 9$, $18 + 9$, $28 + 9$, $38 + 9$, $48 + 9$, etc.
etc. etc. etc. etc. etc.

Although the naming aloud of the addends is not generally permitted, it is tolerated in an oral modification of the foregoing drill, in order to impress upon pupils the fact (without telling them directly) that when they know the first 45 combinations they know everything necessary to enable them to add. These drills consist in having the first pupil say, "5 and 3, 8"; successive ones saying, "15 and 3, 18"; "25 and 3, 28"; etc., etc. Another reason for the use of these combinations is the fact that it gives the pupils an acquaintance with numbers as well as with processes.

Counting drills should be continued, the number to be successively added being the one the pupils find most difficult. This is not necessarily 9, as by degrees they come to note that $55 + 9$ is 1 less than $55 + 10$, but their attention should not be called to this for the present. Counting by 8's, for instance, will take the following form:

(*a*) 1, 9, 17, 25, etc., to 97
(*b*) 2, 10, 18, 26, etc., to 98
(*c*) 3, 11, 19, 27, etc., to 99
(*d*) 4, 12, 20, 28, etc., to 92

and so on, until the beginning number is 8. The final number should not be greater than 99 at this time.

As a preparation for work in multiplication, the counting drills that begin with 0 should receive special attention. A pupil that is asked to add by 7's, for instance, without being given a starting number, says,

"7, 14, 21," etc., and stops at 70.

Another, being told to add by 9's, says,

"9, 18, 27, 36," etc., to 90.

The various exercises prescribed in the preceding grades should be practised in this. The addition examples are gradually made more difficult, the total of each column increasing day by day, until it reaches 29 at the end of the first term, and 39 at the end of the second term. The addends should keep within the prescribed number, and a few examples less difficult than those required by the course should be given occasionally to encourage the slower pupils. It may be advisable to form the latter into a separate division, giving them fewer examples and more time to work each.

Towards the second half of the third year the oral problems should be extended to include the addition of two numbers of two figures each whose sum is less than 100 Only tens should be used. The following are some of the possible combinations:

(*a*) 30+40, (*b*) 20+50, (*c*) 60+20, (*d*) 50+30, (*e*) 40+40

In the special combinations of the first half of the fourth year the addition should not involve reduction (carrying); such as,

(*a*) 32+40, (*b*) 25+50, (*c*) 66+20, (*d*) 57+30, (*e*) 43+40
(*f*) 32+45, (*g*) 25+54, (*h*) 66+22, (*i*) 57+31, (*j*) 43+44

By the end of the year pupils should be required to find sums below 100 of numbers of two figures. The following are types:

(*a*) 32+49, (*b*) 25+57, (*c*) 66+26, (*d*) 57+38, (*e*) 43+49

Problems involving these numbers should not, perhaps, be used as sight work until the pupils have become accustomed to combine them in the proper way. This requires that when the pupil mentally adds 32 and 45, for instance, he thinks 32 and 40 and 5, instead of endeavoring to follow the method pursued in written work, of combining 2 and 5, and then 3 tens and 4 tens. He is tempted, with the numbers in sight, to employ the latter plan. The oral announcement of the numbers by the teacher should require the pupils to use the proper method. When the answers are given orally, the teacher says, "40 and," then after a slight pause "30, Smith," the boy's name following 30 instantly, and an immediate answer being required from Master Smith. If the sums are to be written, the pupils retain their pencils throughout, and the teacher gives the example, "40 and . . . 30, write," without a pause for the comma. In the next stage the pause comes after the tens of the second number, the form being, "42 and thirty . . . four, Jones," or "27 and fifty . . . nine, write." If a pupil attempts, without having the numbers in sight, to add 27 and 59 by first combining 7 and 9 into 16, he frequently forgets the tens' figures. When these are recalled the sum of the ones' figures is forgotten. There is, of course, an occasional child who can retain a mental picture of numbers throughout a long calculation; such a one should be permitted to use his own way, which may be the best for him.

Oral problems containing hundreds may also be introduced in the third grade, beginning with such combinations as (*a*) 500 + 400, (*b*) 300 + 300, (*c*) 600 + 200, (*d*) 200 + 700, the numbers being even hundreds and the totals limited to 900. In the fourth grade the total may exceed 900, typical combinations being the following:

(*a*) 500 + 600, (*b*) 800 + 800, (*c*) 700 + 500, (*d*) 900 + 300

One of the numbers may contain tens also, the sum in such case being less than 1000; as, for instance,

(*a*) 540 + 400, (*b*) 360 + 300, (*c*) 650 + 200, (*d*) 300 + 450

In announcing the sum of 5 hundred and 6 hundred, the pupil should learn to say 11 hundred, without reducing it to 1 thousand 1 hundred.

While oral problems should always be limited to two numbers in the earlier years, a third number may be given occasionally when its introduction does not make the combination more difficult than the grade requires. Thus, in a third grade, an oral example is all right when the numbers to be added are announced, as 9, 37, and 10, or as 37, 10, and 9, but it would belong a year later, perhaps, if the numbers were stated as 9, 10, and 37. In this case the pupils might first combine 9 and 10, making 19, to which they try to add 37. In a higher grade, after pupils have learned that addends may be combined in any order, the last form is useful in a sight example. In stating an oral problem containing three addends, the teacher should pause briefly after the second number to give the pupils an opportunity to combine it with the first; in order that, when the third is stated, the pupils may have only one combination yet to make.

The suggestions already given for the written work of the beginning grades apply in the remaining ones, as far as the addition of integers is concerned. Addition of fractions, of mixed numbers, of decimals, and of compound numbers will be discussed under these respective headings.

Numbers containing dollars and cents, although compound numbers, are not so considered, and their addition follows the methods recommended in the case of ordinary numbers.

In the lesson in which the writing of these numbers is taught, one pupil may be asked to write from dictation several of these numbers in the form of an example in addition. He will probably arrange them in the proper form, dollars under dollars, and cents under cents. In the accompanying example, he may hesitate when "seven dollars" is dictated, but he will probably locate the 7 properly. If he omits the unnecessary ciphers, so much the better, but he may be told to place a decimal point after the 7.

$$\begin{array}{r}\$21.66\\ 9.99\\ 13.50\\ 7.00\\ .65\\ \hline \$\end{array}$$

While some teachers require the writing of two ciphers for the absent cents, in order to fill out each addend, business men prefer a dash. The dollar mark is, of course, omitted before each number except the first and the sum. In the addition of concrete numbers the denomination is abbreviated, and is generally written only with the first number and in the sum. There is as yet no necessity for the teacher to say anything about "like numbers," or to say that cows and wheelbarrows cannot be added.

Oral problems containing numbers composed of both dollars and cents necessarily contain three figures. The first problems of this kind belong in the third year, preferably during the second half. They should be carefully adapted to the maturity of the pupils. Such combinations as (*a*) $1.50 + $.10, (*b*) $2.25 + $.05, etc., are sufficient at this time, although others, apparently more difficult, are readily answered, in sight drills, when arranged in the manner here shown. Numbers containing cents alone, should be limited to the regular combinations of the grade, the sum being less than $1.

$$\begin{array}{r}\$1.50\\ +\ .30\\ \hline\end{array}\qquad\begin{array}{r}\$1.75\\ +\ .25\\ \hline\end{array}$$

Written problems in addition should not be employed to drill pupils in adding. There should be few of them in the lower grades and these should be of the simplest possible character. As the pupils learn subtraction, multiplication, etc., problems involving each new process should be mingled with the others, the numbers being limited to two figures as long as possible, and only a trifle larger than those used in the sight problems. The pupils' ability to determine the required operation is developed more rapidly by problems involving such combinations as the following:

$$(a)\ \begin{array}{r}38\\ \underline{+\ 25}\end{array}\qquad (b)\ \begin{array}{r}38\\ \underline{-\ 25}\end{array}\qquad (c)\ \begin{array}{r}19\\ \underline{\times\ \ \ 4}\end{array}\qquad \begin{array}{l}(d)\ 5\overline{)80}\\ (e)\ \tfrac{1}{4}\text{ of }76\end{array}$$

than it would by those containing larger numbers. As has been already said, consecutive problems should generally involve different operations, and no problem during the first two years should require for its solution more than a single step.

CHAPTER III

SUBTRACTION

In the early examples in written subtraction each figure of the subtrahend is smaller than the corresponding figure of the minuend. As exercises of this kind are very simple, some teachers take up subtraction without reduction ("borrowing") before teaching addition with reduction ("carrying"). Others prefer to postpone the teaching of a second process until the children have given a reasonable time to the first, which is probably a wiser plan.

Unless a different method is prescribed by those in authority, the teacher should avail himself of the pupils' attainments in addition by employing the "building-up" method of subtracting. As a preliminary to written work the teacher gives such oral problems as the following, without stating that they are something new:

(*a*) Mary is 4 years old; in how many years will she be 9?

(*b*) John has 5 cents; how many more must he save to have 8 cents?

These examples are solved by the children through their knowledge of the addition combinations. After a number of such problems have been answered, the teacher may write on the blackboard a few combinations similar to the following:

$$\begin{array}{r} 3 \\ + \\ \hline 5 \end{array} \qquad \begin{array}{r} 5 \\ + \\ \hline 8 \end{array} \qquad \begin{array}{r} 1 \\ + \\ \hline 9 \end{array} \qquad \begin{array}{r} 4 \\ + \\ \hline 5 \end{array} \qquad \begin{array}{r} 2 \\ + \\ \hline 6 \end{array} \qquad \begin{array}{r} 7 \\ + \\ \hline 7 \end{array} \qquad \begin{array}{r} 8 \\ + \\ \hline 9 \end{array}$$

the pupils in turn being required to state the number that should be inserted after the plus sign to make the given total.

The next step begins with a problem, such as the following, to be solved by a pupil at the blackboard:

"If a girl has 76 cherries and gives away 35 of them, how many will she then have?"

The pupil is directed to write 76 and underneath to write 35, placing c. after 76. He is led to say that he must take 35 cherries from 76 cherries to obtain the answer. The teacher shows him that he can do this by saying 5 *and* **1** (writing 1) are 6, 3 *and* **4** (writing 4) are 7. The teacher then places a minus sign before the 35, and tells the children that the name of the sign is *minus*, and that they should write it in front of the smaller number when he tells them to "subtract."

$$\begin{array}{l} 76\text{ c.} \\ \underline{35} \\ 41\text{ c.} \end{array}$$

Examples copied from the blackboard are then worked by the pupils, the size of the numbers depending upon the stage reached in numeration; for example,

$$\begin{array}{rrrrrr} 47 & 39 & 57 & 69 & 98 & 87 \\ \underline{-\ 24} & \underline{-\ 16} & \underline{-\ 20} & \underline{-\ 30} & \underline{-\ 25} & \underline{-\ 44} \end{array}$$

If pupils are acquainted with numbers of three figures, one of them may be sent to the board to subtract 439 from 469, in order to develop the fact that the result is written 30, and not 030. In the early dictated examples the teacher should give the minuend first, saying "From 253 subtract 120," rather than "Subtract 120 from 253," which may be postponed for a later stage.

$$\begin{array}{r} 469 \\ \underline{-439} \end{array}$$

The addition of two numbers should occasionally be given, so that the pupils will feel the necessity of listening to each example, and not assume that one containing two large numbers necessarily indicates subtraction.

The introduction to subtraction with "reduction" may be made by means of examples similar to the following:

$$\begin{array}{rrrrrr} 100 & 124 & 116 & 135 & 153 & 169 \\ -60 & -30 & -45 & -52 & -70 & -84 \\ \hline \end{array}$$

successive pupils being sent to the blackboard to write the results. The attention of the pupils should not be called to the fact that the tens' figure of each minuend is less than the corresponding subtrahend figure. After writing the ones' figure of the answer the first pupil is led to say 6 and **4** (writing 4) are 10; the next to say 3 and **9** (writing 9) are 12; etc.

When many such exercises have been worked by the pupils at their seats, a new lesson should introduce such examples as

$$\begin{array}{rrrrrrr} 50 & 60 & 37 & 42 & 71 & 83 & 90 \\ -25 & -15 & -28 & -16 & -22 & -45 & -86 \\ \hline \end{array}$$

While the pupil selected to work the first example may see at once that the answer is 25, he should be required to say 5 and **5** (writing 5) are 10, 3 and **2** (writing 2) are 5, in order to indicate the procedure required in working the others. By means of the foregoing examples and as many additional ones as are necessary, the class will realize that 1 is to be carried to the second figure of the subtrahend when a number greater than 10 is obtained as the sum of the first figure of the subtrahend and the corresponding figure of the remainder.

As the pupil has in addition always carried 1 to the second column when the sum of the first column is 10, 11, 12, etc., he naturally carries 1 in the new operation, which is another form of addition.

A teacher using the suggested method in subtraction should require his pupils, when they "carry" in addition, to unite the figure carried with the first addend, instead of retaining it until the total of the entire column is obtained.

The next stage introduces examples in which more than one figure of the subtrahend exceeds the corresponding figure of the minuend. The size of the numbers will depend on the point reached in numeration and notation.

Examples are then dictated, some in the addition of two numbers, and others in subtraction. The wording of a subtraction example may be changed, in a higher grade, to this form: "Take 3750 from 60,000," in which the subtrahend is read first, or, "Find the difference between 1806 and 10,090." A few examples may be worked on the blackboard in which the subtrahend is written above the minuend:

$$\begin{array}{r} -\,2{,}057 \\ \underline{10{,}863} \end{array} \qquad\qquad \begin{array}{r} -\,6{,}954 \\ \underline{23{,}006} \end{array}$$

the pupils not being permitted to rewrite the examples with the order of the terms changed.

During the fourth year, such examples as the following may occasionally be copied from the blackboard and the results obtained:

$$\begin{array}{r} 39 \\ \underline{+\ ?} \\ 100 \end{array} \qquad \begin{array}{r} ? \\ \underline{+\ 23} \\ 86 \end{array} \qquad \begin{array}{r} 134 \\ \underline{-\ 25} \\ ? \end{array} \qquad \begin{array}{r} 127 \\ \underline{-\ \ ?} \\ 56 \end{array} \qquad \begin{array}{r} ? \\ \underline{-103} \\ 250 \end{array}$$

In these the sign does not necessarily indicate the required operation. During this year an occasional blackboard exercise in the following form is presented, in which the work is done without rewriting the numbers.

$$\begin{array}{rcrcl} 116 & - & 37 & = & a \\ 250 & - & 116 & = & b \\ 307 & - & 298 & = & c \\ \hline d & - & e & = & f \end{array}$$

The remainders represented by a, b, and c are first obtained. Their total, f, should be equal to the difference between d, the total of the first column, and e, the total of the second.

The recommendation that written exercises in subtraction correspond with those in addition as regards the size of the figures does not mean that the subtrahend should contain as many figures as the minuend. The former should frequently contain only two or three figures when the minuend contains five or four.

Pupils should prove every subtraction result by covering the minuend, combining the remainder with the subtrahend, adding upwards, and comparing the result with the given minuend.

When it is necessary to mention the subtraction sign the teacher should call it by its name. Thus he speaks of 48 − 19 as "48 minus 19," not employing the form affected by some: "48 take away 19." Pupils learn the necessary technical terms by hearing the teacher say them, and by being required to use them when they should be used. No definition of any of them is needed, and none should be memorized. Technical terms should not be introduced unnecessarily; notation and numeration, for instance, need not be mentioned during the first four years. The word "addend" is, perhaps, not needed; "subtrahend," "minuend," and "remainder" are required; "difference" may be used in a problem without telling the pupils that the remainder in subtraction is also called the difference. Some teachers require their pupils to label the several numbers as "subtrahend," "remainder," etc. This is a waste of time; "Ans." after the result is sufficient.

Subtraction of United States money is taught just after the pupils have learned to add numbers containing dollars and cents. The early minuends should seldom exceed numbers of four figures. When they consist of dollars alone, children should be discouraged from writing unnecessary ciphers in the cents' places. They learn almost of themselves to locate the decimal point properly.

$$\begin{array}{r} \$10.\,- \\ \underline{1.58} \\ \$ \end{array} \quad \textit{Ans.}$$

While the sight drills in subtraction should cover the 45 addition combinations, the pupils' acquaintance with the latter makes it unnecessary to dwell upon the simple ones, or to take them up in any particular order. The first series may consist of those in which the minuend contains numbers below 10; such as,

$$\begin{array}{rrrrrrrrrr} 9 & 8 & 7 & 8 & 9 & 6 & 5 & 6 & 7 & 8 \\ -\,2 & -\,4 & -\,2 & -\,3 & -\,5 & -\,2 & -\,2 & -\,3 & -\,5 & -\,2 \\ \hline \end{array}$$

the sign being essential, especially when addition combinations containing the same numbers are also used in review exercises; such as,

$$\begin{array}{rrrrrrrrrr} 9 & 8 & 7 & 8 & 9 & 6 & 5 & 6 & 7 & 8 \\ +\,2 & +\,4 & +\,2 & +\,3 & +\,5 & +\,2 & +\,2 & +\,3 & +\,5 & +\,2 \\ \hline \end{array}$$

The second series consists of those in which the minuend contains numbers from 10 to 18, the subtrahend and the remainder in each case being less than 10; such as,

$$\begin{array}{rrrrrrrrr} 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ -\,2 & -\,5 & -\,9 & -\,4 & -\,6 & -\,7 & -\,9 & -\,8 & -\,9 \\ \hline \end{array}$$

These should be very numerous, as it is all-important to have pupils give remainders almost automatically. The accompanying addition drills will be the more difficult combinations already used, with the sign (+) prefixed to the second addend. The review of the addition combinations is fully as important as the subtraction drills, especially when the Austrian method is employed.

The other subtraction drills in finding differences between numbers less than 20 should be placed in a horizontal line, as pupils might be tempted to follow the written method of beginning with the ones' column if the numbers were arranged as they are in the earlier drills. These, being intended as an introduction to written work in a process, follow the arrangement of the latter; the other drills, which are used to develop a pupil's knowledge of number, should be so arranged as to prevent the improper use of process methods. The following are suggested types:

18 − 5, 18 − 13; 19 − 2, 19 − 17; 17 − 6, 17 − 11
etc. etc. etc.

each set of two examples having the same minuend, and the subtrahend of one being the remainder of the other. An addition combination consisting of two different numbers has two corresponding forms in subtraction, as shown above, both of which should be used in drill work and in oral problems. Thus, 49 + 8 gives rise to 57 − 49 and 57 − 8, etc. All of these latter drills are written in this form.

Even when pupils employ the building-up method in written subtraction, they tend naturally, but insensibly, to the use of the other plan in many of the later examples. Thus, a pupil who might in 18 − 17 think 17 and 1 are 18 is less likely to continue to do so in 18 − 1. The same is true of 57 − 49 and 57 − 8, in the former of which his addition work leads him to think 49 and 8. In the latter, the addition columns do not present the combination 8 and 49. When the tendency to "take away" 2 from 19, for instance, appears in a pupil's analysis of an oral problem, no objection should be made.

The teacher must remember that oral problems are employed for several distinct purposes; one of which is to teach number, another being the training of the pupil to determine the operation required. They are not intended to show how processes are performed; in fact, they frequently require a different procedure. The oral addition of 37 and 50 should be begun at the tens' figures, for instance; $20\frac{1}{2}$ is subtracted from 50 by first deducting 20, and then $\frac{1}{2}$, etc.

In the most common type of written problems, the pupil will see that he has to take one number from another, but when he has determined that subtraction is the required operation, he performs it by "building-up" and not by "taking away." The teacher, therefore, need have no fear of confusing his pupils when he suggests the deduction of 30 from 85 and of 7 from the remainder in the oral exercise 85 — 37. This is done through the previous drills, which are similar to those in addition for the corresponding grades:

$$80 - 30 \qquad 85 - 30 \qquad 85 - 37$$

This plan yields the answer in a better form than does the other, whereby the pupil thinks 37 and 40 are 77, and 8 are 85, in which the two portions of the result, 40 and 8, are separated by 77, and their combination must be made after the introduction of 85. By the one suggested, the pupil begins with 85, then thinks "less 30 (55), less 7 (48)," obtaining the results directly.

Exercises in counting forwards are intended to accustom children to the several successive steps required in the addition of a long column. Since in each subtraction example there is but one step to each column, which is covered by the ordinary drills in finding the difference between two numbers, there should be no exercises in counting backward.

While the preliminary oral problems in subtraction are of such a character as to require pupils to go forward from one given number to the other, most of the later problems, oral and written, should be of the other form.

The following are oral problems of the first type:

(*a*) Mary has 18 rose bushes in her garden; how many must she plant to have 25?

(*b*) A boy has to ride 15 miles on his bicycle; how far has he to go after he has ridden 9 miles?

The second type includes problems like the following:

(*c*) A farmer had 60 tons of hay. How many remained after he had sold 55 tons?

(*d*) A woman gave a 50-cent piece to pay for a 5-cent cake of soap; how much change did she receive?

Until the latter half of the third year, or thereabouts, written problems in subtraction should be limited to numbers of two figures, the same limitation being applied to addition problems, which should generally contain only two numbers. The warning that written problems are not intended to give children facility in performing processes cannot be too often repeated; this should be obtained through the abstract work. Written problems are designed to train children to determine of themselves through the conditions in each case what operation is required. Those who readily solve oral problems frequently have difficulty with written ones, especially when the latter contain large numbers.

Real problem work begins with the introduction of those involving subtraction. Up to this time the children need pay no attention to the wording, the problem is necessarily one in addition. To require them to listen to the conditions, a problem should contain but two numbers, since three would indicate addition; and such a word as "left" or "remainder" should not be employed to suggest that subtraction is called for. When a problem is stated orally by the teacher, he should read it twice, the first reading being intended to give the pupils an opportunity to give particular attention to the conditions. At the second reading, they take their pencils, writing each number as it is spoken. The required operation is then indicated by the proper sign and the solution follows. In a subtraction problem the minuend should be read first, for some time. In the next term, the subtrahend, say 29 cows, is occasionally read first, and the minuend, 56 cows, next. The pupil will generally write the latter above the former; but if he happens to place it underneath, he should write the minus sign in front of the subtrahend, and proceed with the work.

$$\begin{array}{r} -29\phantom{\text{ c.}} \\ \underline{56\text{ c.}} \\ \text{c.} \end{array}$$

A problem in addition should not always alternate with one in subtraction, as this might enable a pupil to determine that a given problem involved addition, since the previous one was in subtraction. When pupils solve problems from their books, the latter should not contain answers.

The employment of large numbers in problems comes with the introduction of United States money, but they need be very little larger than those used in sight combinations. The following are sufficiently difficult for the earlier problems:

$$\begin{array}{rrrrr} \$1.57 & \$1.57 & \$1.75 & \$.79 & \$2.95 \\ \underline{+\ .39} & \underline{-\ .39} & \underline{-\ 1.57} & \underline{+\ .98} & \underline{-\ 2.46} \end{array}$$

The numbers employed in the oral problems should correspond with those recommended for addition; such as,

$$\$1.60 - \$1.50,\ \$1.60 - \$.10,\ \$2.30 - \$.05,\ \$2.30 - \$2.25$$

In sight drills, an occasional combination, presented in the following form, may be tried in the second half of the third year:

$$\begin{array}{rrrr} \$1.75 & \$2.\text{—} & \$1.75 & \$2.\text{—} \\ \underline{-\ .50} & \underline{-\ .25} & \underline{-\ 1.25} & \underline{-\ 1.75} \end{array}$$

At this time are begun problems, oral and written, involving two steps. These may include any two combinations, provided that throughout the entire work no number of more than two figures appears until the fourth year. Combinations of the following types may occur in the written problems:

$$\begin{array}{lll} 35 + 25 - 14 & 70 - 17 \times 3 & (26 + 19) \times 2 \\ 70 - 25 - 16 & 19 \times 2 \times 2 & (18 + 28) \div 2 \\ 90 - (37 + 28) & 81 \div 3 \div 3 & 26 + 19 \times 2 \end{array}$$

Those in the oral problems should contain simpler numbers; as:

$$\begin{array}{lll} 30 + 6 - 10 & 50 - 9 \times 5 & (16 + 4) \div 2 \\ 90 - (75 + 8) & 27 \div 3 \div 3 & 5 + 6 \times 2 \\ 70 - 8 + 10 & 6 \times 2 \times 2 & (5 + 6) \times 2 \end{array}$$

The signs shown above are used merely to indicate to the teacher the character of the operations. As a rule they should not be used in abstract examples given to pupils before the fifth year. When a child wishes to indicate the operations required to solve the foregoing written problems, he may do so in some such way as this:

$$\begin{array}{rrrrrr} 35 & 70 & 19 & 26 & 18 & \\ \underline{+\ 25} & \underline{-\ 25} & \underline{\times\ 2} & \underline{+\ 19} & \underline{+\ 28} & 3\overline{)81} \\ * & * & * & * & 2\underline{)\ *} & 3\underline{)\ *} \\ \underline{-\ 14} & \underline{-\ 16} & \underline{\times\ 2} & \underline{\times\ 2} & \textit{Ans.} & \textit{Ans.} \\ \textit{Ans.} & \textit{Ans.} & \textit{Ans.} & \textit{Ans.} & & \end{array}$$

This arrangement provides a place (*) in which to write the result of the first step, and below it is written the number by which it is to be increased (+), diminished (—), multiplied (×), or divided (÷). The remaining three problems require a somewhat different arrangement; as follows:

$$\begin{array}{rrcrrcrr} 37 & 90 & & 17 & 70 & & 19 & 26 \\ \underline{+\ 28} & \underline{-\ \ *} & & \underline{\times\ \ 3} & \underline{-\ \ *} & & \underline{\times\ \ 2} & \underline{+\ \ *} \\ \multicolumn{2}{c}{\textit{Ans.}} & & \multicolumn{2}{c}{\textit{Ans.}} & & \multicolumn{2}{c}{\textit{Ans.}} \end{array}$$

The work at the blackboard should develop several forms, from which different pupils may select those that suit their needs. It is important that pupils should indicate the successive steps when they write the numbers during the teacher's second reading of a problem.

Sight drills should be limited to those of one operation.

For the information of teachers, brief descriptions of other methods of performing subtraction are given herewith. It is understood, of course, that pupils are to learn but one way.

The method suggested at the beginning of this chapter is called by various names; the Austrian method, the Continental method, the shop method, the building-up method, the method of complementary additions, etc. It is much used in Germany, and is gradually coming into favor in this country.

The one favored by teachers who believe that children should understand the reason for each step in an operation is the method of decomposition. Thus, the difference between 5030 and 268 is obtained by decomposing these numbers in the manner shown below:

$$\begin{array}{rcrcrcrcr}
5030 = 5000 + 30 & = & 4000 & + & 900 & + & 120 & + & 10 \\
-\ 268 & = & & - & 200 & - & 60 & - & 8 \\
\hline
4762\ \textit{Ans.} & = & 4000 & + & 700 & + & 60 & + & 2
\end{array}$$

These numbers are not written in this way by the children, their work taking the following form if they are not forbidden to use the cancellation marks and the additional figures. The "explanation" is made somewhat as follows: "8 ones from 0 ones I cannot take, so I borrow 1 ten from 3 tens, making the latter 2 tens, and the 0 ones 10 ones; 8 ones from 10 ones leaves 2 ones,

$$\begin{array}{cccc}
 & 9 & 12 & \\
4 & \cancel{10} & \cancel{2} & 10 \\
\cancel{5} & \cancel{0} & \cancel{3} & \cancel{0} \\
- & 2 & 6 & 8 \\
\hline
\end{array}$$

which I write," etc.; the same wordy statements being required for the successive numbers of the subtrahend. The introduction of single splints, bundles of 10 splints, packages of 100 splints consisting of 10 bundles of 10 each, etc., are recommended as an assistance in clarifying the procedure. When the pupils are permitted to write the new figures over those of the original example, as shown on page 74, the work is rendered less difficult for the beginner, but it is not understood.

Some teachers, disliking the word "borrow," modify the phraseology to suit the separation of the minuend into the various portions. Some writers on methods recommend the employment of 5030 splints arranged in 5 boxes of 1000 each and 3 bundles of 10 each to illustrate the decomposing process, etc.

The method of equal additions increases, when necessary, any figure of the minuend; and it preserves the correctness of the result by increasing the subtrahend to the same extent. The procedure, which is different from that followed in the preceding method, is shown in the subtraction of 268 from 5030.

The pupil beginning at the ones' column finds 8 of the subtrahend greater than 0 in the minuend. He therefore increases the latter by 10, and from this takes 8, leaving 2. To preserve the proper relation between the two terms he increases the subtrahend by 1 ten, making the next figure 7 (tens). As this is greater than 3 (tens) of the minuend, he increases the latter by 10 (tens), making it 13 (tens), from which 7 (tens) is taken, leaving 6 (tens). The subtrahend is now increased by 10 tens or 1 hundred, which makes 3 (hundreds) in the subtrahend; etc.; etc. The accompanying example shows

$$\begin{array}{cccc} & 10 & 13 & 10 \\ 5 & \cancel{0} & \cancel{3} & \cancel{0} \\ 1 & 3 & 7 & \\ & \cancel{2} & \cancel{6} & 8 \\ \hline 4 & 7 & 6 & 2 \end{array}\ \textit{Ans.}$$

the increase of each of the three last figures of the minuend by 10 ones, 10 tens, and 10 hundreds, and an equivalent increase of the proper subtrahend figures by 1 ten, 1 hundred, and 1 thousand.

This may be arranged as follows:

$$\begin{array}{rcl} 5030 + 10\text{ h.} + 10\text{ t.} + 10 & = & 5\text{ th.} + 10\text{ h.} + 13\text{ t.} + 10 \\ 268 + 1\text{ th.} + 1\text{ h.} + 1\text{ t.} & = & \underline{1\text{ th.} + 3\text{ h.} + 7\text{ t.} + 8} \\ \text{The difference being} & & 4\text{ th.} + 7\text{ h.} + 6\text{ t.} + 2 \end{array}$$

or, 4762 *Ans.*

This method is less difficult than the preceding one, as it can be readily used without the writing of the extra figures. As the pupil finds 8 greater than 0, he says "8 from 10 leaves 2"; he then increases 6 by 1, and says "7 from 13 leaves 6," having increased 3 tens by 10 tens; etc. The saying of a minuend combination greater than 10 reminds the pupil that he should carry 1 to the next figure of the subtrahend. He makes no change in a minuend figure other than the prefixing of 1 when necessary, which is simpler than remembering to decrease the minuend figure by 1 at the proper time and then to increase it, when required, by 10.

Subtraction is taught in this way in England by teachers who believe that explanations are unwise, and who nevertheless prefer this method to the Austrian method because they think that subtraction should be made as different as possible from addition. The method was formerly in vogue in this country until teachers were led to believe that the child should be asked to take no step without fully understanding it, and as this method did not lend itself to an apparently simple form of explanation, it was largely displaced by the method of decomposition.

The merit of the Austrian method consists in taking full advantage of the previously-acquired knowledge of the pupil. When he takes 8 from 15 in the other two methods his mind naturally inquires for the number that makes 15 when he adds it to 8; and he must then reason that 8 from 15 gives the same result, 7. In the Austrian method he is saved this second step.

Some teachers that believe in the Austrian method do not use it properly. They teach a boy to say 8 and **2** (writing 2) are 10, 6 and **6** (writing 6) are 12, changing the 3 tens of the minuend to 2 tens to repay the borrowed 10 ones. The real method assumes that the minuend 5030 corresponds to the sum in an addition example of which 268 is one addend, and that the other addend may be obtained by writing below (or above) 268 the figures required to produce 5030. The ones' figure is **2**, obtained by saying 8 and **2** are 10. Since in addition 1 (ten) is carried to 6 (tens) to make 7 (tens), the same should be done in subtracting by this method. It is the natural thing for the child to carry 1 after obtaining 10, and a different procedure is confusing, especially as it requires him to change a figure in the number, 5030, which is considered as the sum. In later applications of this method the carrying figure may be 5, 7, etc., as will be shown later on.

$$\begin{array}{r} 5030 \\ -\ 268 \\ \hline 2 \end{array}$$

$$\begin{array}{r} 5030 \\ 268 \\ \hline 2 \end{array}$$

$$\begin{array}{r} +10 \\ 5\ 0\ 3\ 0 \\ 2\ 6\ 8 \\ \hline \end{array}$$

A modification of the other methods consists in subtracting 6 from 13 by first deducting it from 10 and adding 3 to this remainder.

Even in the illustration of the earliest examples by means of objects, the extreme method-worshiper considers that two types should be distinguished. To find the *remainder* left after taking 6 from 10, 10 counters are employed, 6 of which are slightly separated from the others.

0 0 0 0 0 0 0 0 0 0

To ascertain the *difference* between 10 and 6 they claim 16 counters should be so arranged as to show how many counters of the larger group extend beyond the smaller group. The latter operation is sometimes called comparison, etc., to distinguish it from the former, which is considered real subtraction.

0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0

Under no circumstances should children be confused by two forms of subtraction. Subtraction taught by the "building-up" method to pupils able to add, does not require the use of objects.

CHAPTER IV

MULTIPLICATION

In schools that teach addition without reduction in the second term of the first year, addition with reduction and subtraction without reduction are frequently taught during the first half of the second year; in the latter half, subtraction with reduction and multiplication without reduction being taken up. This preliminary work in multiplication limits the partial products to numbers below 10. When 2 is the multiplier, the multiplicand is a number composed of digits less than 5; when the multiplier is 3, each digit in the multiplicand is less than 4; when 4 is the multiplier, each digit in the multiplicand is less than 3. In the case of larger multipliers, the digits in the multiplicand are 1's or 0's. The following examples are illustrations:

$$\begin{array}{lrlrlrlr} (a) & 210 & (b) & 101 & (c) & 321 & (d) & 234 \\ & \underline{\times 4} & & \underline{\times 5} & & \underline{\times 3} & & \underline{\times 2} \end{array}$$

$$\begin{array}{lrlrlrlr} (e) & 111 & (f) & 110 & (g) & 101 & (h) & 111 \\ & \underline{\times 6} & & \underline{\times 7} & & \underline{\times 9} & & \underline{\times 8} \end{array}$$

The oral work in schools using the modified Grube system is commenced as early as the first year with the finding of products to 9, the work during the second year extending to larger numbers. The better plan postpones the introduction of oral work in multiplication until the third year, written work being commenced at the same time.

The teacher should not be misled by the text-book into requiring his children to begin by learning all of the tables, or even the table of 2's. The work is started by oral problems involving the use of 2 as a multiplier; such as,

(*a*) What is the cost of 2 melons at 5 cents each?

(*b*) How many panes of glass are there in 2 windows, if each contains 8 panes?

(*c*) Jane is 6 years old. Her brother is twice as old. How old is her brother?

These should include all multiplicands to 9, and should involve conditions with which the children are familiar. It makes no difference at first how the children obtain the results; they probably will add 5 cents and 5 cents in (*a*), 8 panes and 8 panes in (*b*), etc., but the analysis, if used, should say, "2 melons will cost twice 5 cents, or 10 cents."

Unless the course of study requires that written multiplication be commenced by the use of examples that do not involve reduction ("carrying"), a pupil may be sent to the board to obtain the answer to the following problem: When dolls are 25 cents each, what will 2 dolls cost?

Although the pupil may desire to give the answer at once, he is asked to write out his work on the blackboard. If he employs addition, the teacher may tell him that there is another way; that he can write 25 cents, underneath it write 2, and say twice 5, etc. The pupil should be given a reasonable time to obtain for himself the product, 50¢. Other pupils are then sent to the board to work such abstract examples as the following:

$$\begin{array}{r} 25\text{¢} \\ +\ 25\text{¢} \\ \hline 50\text{¢} \end{array} \qquad \begin{array}{r} 25\text{¢} \\ \times\ 2 \\ \hline 50\text{¢} \end{array}$$

(*a*) $\begin{array}{r} 45 \\ \times 2 \\ \hline \end{array}$ (*b*) $\begin{array}{r} 51 \\ \times 2 \\ \hline \end{array}$ (*c*) $\begin{array}{r} 234 \\ \times 2 \\ \hline \end{array}$ (*d*) $\begin{array}{r} 127 \\ \times 2 \\ \hline \end{array}$ (*e*) $\begin{array}{r} 306 \\ \times 2 \\ \hline \end{array}$

The method of reading the foregoing examples must be determined by the teacher, whether to call (*a*) 2 times 45 or 45 multiplied by 2. If he selects the first method, the tables will take the following form. They should first be made on the board by successive pupils, each of whom writes one line, the teacher showing how the factors are to be arranged in the first few lines, and leaving the children to determine and to write the products. Each successive table is developed first on the blackboard, the children writing it later on paper.

$2 \times 1 = 2$
$2 \times 2 =$
$2 \times 3 =$
$2 \times 4 =$
$2 \times 5 =$
$2 \times 6 =$
$2 \times 7 =$
$2 \times 8 =$
$2 \times 9 =$

Sight drills of various kinds are employed with 2 as the multiplier, the arrangement of the factors following the form selected by the teacher; those preferring to read × as "multiplied by," place 2 after the sign. The reason given for this preference is the fact that ÷ is always read "divided by," and it is deemed advisable to read these related signs in the same manner.

Teachers also disagree as to the form in which the multiplication table should be taught. While many employ the one given above, others prefer to make 2 the second term in its table, as is here shown. When, however, 2 is to be used for some little time as a multiplier, the first form is more useful.

Once 2 is 2
Twice 2 are 4
3 times 2 are 6
4 times 2 are 8
etc.

After the class has had sufficient practice in multiplying by 2 in written work, a pupil is sent to the board to multiply 222 by 3, no explanation being given; another to multiply 212 by 4; another to multiply 122 by 6, etc., any number to 9 inclusive being used as a multiplier, but the multiplicands being limited to numbers containing only 0, 1, and 2.

The sight drills at this stage will then include such combinations as:

$$\begin{array}{rrrrrrrr} 8 & 2 & 7 & 2 & 6 & 2 & 5 & 2 \\ \underline{\times 2} & \underline{\times 3} & \underline{\times 2} & \underline{\times 4} & \underline{\times 2} & \underline{\times 9} & \underline{\times 2} & \underline{\times 7} \\ \end{array}$$

etc. etc. etc. etc.

one factor in each set being 2.

Oral problems may then be given in which 2 is the multiplicand and 6, 7, 8, etc., are the multipliers.

At the beginning it is well to progress slowly, many abstract examples being given with 2 as a multiplier before regular work with 3 is taken up.

Oral work and written work with 3 as a multiplier are commenced without preliminary treatment, the first multiplicands being limited to numbers containing only 0, 1, 2, and 3. The use of the large multipliers in the incidental examples and the informal introduction of the new multiplier 3 give a sense of power, which the teacher will do well to induce as frequently as possible. The multiplicands may bring in larger digits by degrees or even all of them in one lesson, by such examples as the following, the teacher not expecting the results too rapidly:

$$\begin{array}{rrrrrr} (a)\ 44 & (b)\ 55 & (c)\ 66 & (d)\ 77 & (e)\ 88 & (f)\ 99 \\ \underline{\times 3} & \underline{\times 3} & \underline{\times 3} & \underline{\times 3} & \underline{\times 3} & \underline{\times 3} \end{array}$$

$$\begin{array}{rrrrrr} (g)\ 444 & (h)\ 555 & (i)\ 666 & (j)\ 777 & (k)\ 888 & (l)\ 999 \\ \underline{\times 3} & \underline{\times 3} & \underline{\times 3} & \underline{\times 3} & \underline{\times 3} & \underline{\times 3} \end{array}$$

The majority of the pupils will obtain the products by adding 4 + 4 + 4, 5 + 5 + 5, etc., in this way learning how the table is formed.

This may require a few days; in the meantime examples may be given, having such multiplicands as:

43, 54, 65, 76, 87, 98,
432, 543, 654, 765, 876, 987

the successive digits to the left increasing by 1. When the table is formed by the pupils and memorized, the multiplicand may be any number whose product does not contain more figures than the grade work calls for.

Although concert work should be forbidden in most cases, it may be usefully employed in memorizing the tables. A preliminary step to the memorizing of a new table is a counting exercise; that for the 4-times table being 4, 8, 12, 16, etc., to 36, it being unnecessary at this stage to go beyond 4 times 9. This fixes the products and gives the pupils a further insight into the formation of the tables. They may then for seat work write out each table at the proper time in the regular form, and memorize it.

As a continuation of the work with 3, and before proceeding to the employment of 4 as the regular multiplier, examples are given in which 4, 5, 6, 7, 8, and 9 are used as multipliers, the multiplicand in each case being limited to numbers containing only 0, 1, 2, and 3. The first examples may be the following:

$$(a)\ \begin{array}{r}321\\ \times 4\\ \hline\end{array}\quad (b)\ \begin{array}{r}321\\ \times 5\\ \hline\end{array}\quad (c)\ \begin{array}{r}321\\ \times 6\\ \hline\end{array}\quad (d)\ \begin{array}{r}321\\ \times 7\\ \hline\end{array}\quad (e)\ \begin{array}{r}321\\ \times 8\\ \hline\end{array}\quad (f)\ \begin{array}{r}321\\ \times 9\\ \hline\end{array}$$

Later multiplicands employ the same figures in different combinations. When the teacher employs cards for drills he may add the following to those already in use:

$$(a)\ \begin{array}{r}3\\ \times 3\\ \hline\end{array}\quad \begin{array}{r}4\\ \times 3\\ \hline\end{array}\quad \begin{array}{r}5\\ \times 3\\ \hline\end{array}\quad \begin{array}{r}6\\ \times 3\\ \hline\end{array}\quad \begin{array}{r}7\\ \times 3\\ \hline\end{array}\quad \begin{array}{r}8\\ \times 3\\ \hline\end{array}\quad \begin{array}{r}9\\ \times 3\\ \hline\end{array}$$

$$(b)\ \begin{array}{r}3\\ \times 4\\ \hline\end{array}\quad \begin{array}{r}3\\ \times 5\\ \hline\end{array}\quad \begin{array}{r}3\\ \times 6\\ \hline\end{array}\quad \begin{array}{r}3\\ \times 7\\ \hline\end{array}\quad \begin{array}{r}3\\ \times 8\\ \hline\end{array}\quad \begin{array}{r}3\\ \times 9\\ \hline\end{array}$$

While the successive tables grow more difficult, the number of new combinations decreases with each. In the

3-times table there are seven new ones; in the 4-times table, six; in the 5-times table, five, etc. When the pupils reach the table of 9's, they have already learned, in another form, the product of 9 by 1, 2, 3, 4, 5, 6, 7, and 8, respectively, which leaves but a single new one, 9 times 9.

There are in multiplication, as in addition, only forty-five possible combinations of two numbers of one figure each. Omitting the products by 1, there are but thirty-six, and the drills should secure the mastery of all of them.

When a pupil seems utterly unable to memorize a table, he should be permitted to keep before him, while at work, one he has himself made. It is very important that he should learn how multiplication is performed, even though he cannot do it as rapidly as his mates or in the customary way. Other members of the class should not, however, be permitted to use any unnecessary aid, such as carrying figures, etc.

As a preliminary to written multiplication by multipliers of two digits, sight exercises are had in multiplying by 10. After these come multipliers consisting of tens; such as, 20, 30, etc., to 90. A pupil is asked to multiply, for instance, 43 by 20. As he knows that 43 multiplied by 10 is 43 tens, or 43 followed by a cipher, he is easily led to see that 43 multiplied by 2 tens gives as the product twice 43 tens, or 86 tens, or 86 followed by a cipher. Other pupils, in turn, multiply 46, 54, 65, etc., by 20, 30, 40, etc., the fact being developed that the work begins by using 0 as the first multiplier, its product, 0, being written under it; and that the first figure, 6, of the product by 2 tens, is written under the 2.

$$\begin{array}{r} 43 \\ \times 20 \\ \hline 860 \end{array}$$

Sight drills may be had which involve no carrying, the products not exceeding 990; such as:

(*a*) 42 × 20 (*b*) 33 × 30 (*c*) 21 × 40 (*d*) 34 × 20

$$\begin{array}{r} 32 \\ \times 21 \\ \hline 32 \\ 64\cancel{0} \\ \hline 672 \end{array} \ \textit{Ans.}$$

The method of multiplying by a number composed of two significant figures (a number containing no cipher) may be illustrated on the blackboard by a pupil, who is told to multiply 32 by 21. The teacher, as usual, should say nothing until the pupil needs assistance. If he begins by using 2 tens as the first multiplier, or if he places the right-hand figure of both partial products in the same column, he should be permitted to proceed to the end and to obtain 96 as the result. In the latter case, he should be asked by what number 32 is multiplied when 96 is the product. The fact that 20 times 32 is 640, and 21 times 32 is 640 plus 32, may be brought out, etc. The pupil who realizes the importance of examining his answer has learned a useful lesson. The method is finally developed that the product by 1 is first written, its right-hand figure being placed under 1, and that the right-hand figure of the product by 2 (tens) is placed under 2. If the pupil places a cipher after 64, the second partial product, the fact that it is unnecessary may be shown, and that the place value of 2 need not be considered if its partial product is located according to the method given above. As has already been stated, the value of the Arabic system is due to the fact that it renders unnecessary any attention during a calculation to the place value of the different digits. The teacher should not unnecessarily confuse a child by dwelling on this point, although an occasional reference to tens, hundreds, etc., when required, need not be avoided.

$$\begin{array}{r} 46 \\ \times 23 \\ \hline 92 \\ 138 \\ \hline 1058 \end{array} \ \textit{Ans.}$$

English teachers favor working a multiplication example by commencing with the left-hand digit of the multiplier; but the general practice in this country is to start with the ones' figure, thus beginning the operation at the right as in addition and subtraction. In the upper grades, pupils should be shown that the figures of the multiplier may be used in any order. The advocates of the English plan think it helps in division to accustom children to consider the multiplier 21 as twenty and one, instead of one and twenty, etc.

The early multipliers should be composed of 1's, 2's, 3's, etc., rather than of 8's and 9's; such as 21, 31, 22, 13, 23, 32, etc. The multiplicands should contain but two figures at first, and the products should not exceed the limits prescribed by the course of study for the grade.

The use of 3-figure multipliers presents nothing new. A pupil will probably work such an example at the board without making a mistake. If he errs in locating the third partial product, 730, some member of the class will, doubtless, be able to show where the cipher should be placed.

$$\begin{array}{rl} 365 & \\ \times 234 & \\ \hline 1460 & \\ 1095 & \\ 730 & \\ \hline 85410 & \textit{Ans.} \end{array}$$

By the end of the fourth year, the multiplier should contain a cipher. In multiplying by 240, the pupil at the board may neglect to insert the final cipher in the first partial product, especially as the product by 4 produces a terminal cipher. His attention should be called to the fact that the product of 365 by 0 is 0, that the right-hand figure of the product by 4 (tens) is placed under 4, which is written just before the other cipher in the same line.

$$\begin{array}{rl} 365 & \\ \times 240 & \\ \hline 14600 & \\ 730 & \\ \hline 87600 & \textit{Ans.} \end{array}$$

In multiplying 365 by 204, only two lines of partial products should be written, corresponding with the number of significant figures in the multiplier. While some teachers favor a row of ciphers to indicate the product of the multiplicand by 0, it is altogether unnecessary. In this as in all other calculations, care in the arrangement of the figures eliminates one source of error.

$$\begin{array}{rl} 365 & \\ \times 204 & \\ \hline 1460 & \\ 730 & \\ \hline 74460 & \textit{Ans.} \end{array}$$

While the natural test of the correctness of a product is its division by the multiplier, which produces the multiplicand when all the work is correct, early examples in multiplication cannot be tested in this way, as pupils learn division after they have been taught to multiply. Working the example a second time is not altogether satisfactory, as a pupil who thinks 7 times 8 are 57 will repeat his mistake. A product by 4 may be tested by multiplying twice by 2; a product by 6, by multiplying by 3 and multiplying this product by 2; etc. A product by 3 may be tested by adding the multiplicand to twice the multiplicand; one by 5, by adding the multiplicand to 4 times the multiplicand. In making a test, the first result should not be looked at until the second is obtained. The product of two factors of two or three figures each is tested by reversing the order of the factors.

The multiplication drills follow the prescribed oral work, those for the first half of the second year covering products to 9 × 5 and 5 × 9, which are extended during the next half to 9 × 9, and in the first half of the third year to 12 × 12.

By the end of the fourth year, pupils should be able to give at sight any product of two numbers which does not exceed 100, the development of the work following that for addition.

3d year, 2d half, 30 × 3, 2 × 40, 20 × 4, etc.
4th year, 1st half, 31 × 3, 2 × 44, 22 × 4, etc.
4th year, 2d half, 16 × 6, 7 × 14, 23 × 4, etc.

The oral problems of these three terms may include the larger numbers suggested for addition. The following combinations are typical:

3d year, 2d half, 300 × 3, 2 × 400, 200 × 4, etc.
4th year, 1st half, 300 × 4, 2 × 700, 600 × 7, etc.
4th year, 2d half, 310 × 3, 2 × 440, 210 × 4, etc.

Some sight drills have already been suggested in which the multipliers consist of tens.

Written problems in multiplication should contain small numbers, products to the middle of the third year not exceeding 100. One-step problems may have larger products during the latter half of the third year, but those involving two operations should be limited to the types given on page 73. Nothing is gained at this stage by tedious or complicated problems.

It will probably be necessary for the teacher to prepare the early problems. In the oral ones employed to teach the number combinations, the answer to a problem asking the number of panes in 4 windows each containing 8 panes is the same as that obtained to one requiring the number of panes in 8 windows each containing 4 panes. Neither of these problems would serve as an introduction to the first written problem, as the multiplicand should exceed 12, and windows seldom have so many panes. Children in the habit of going to the store may be given a written problem that involves the cost of 2 articles at 17 cents each, for instance, the written solution of which takes the accompanying form.

$$\begin{array}{r} 17\ \text{¢} \\ \times\ \ 2\phantom{\ \text{¢}} \\ \hline 34\ \text{¢}\ \textit{Ans.} \end{array}$$

$$\begin{array}{r} \$\ .17 \\ \times\ \ \ 2 \\ \hline \$\ .34 \end{array}$$

When, however, addition and subtraction of United States money have already been taught, the decimal point introduces a feature that makes it inadvisable to use cents in the first written problems.

The number of desks in 2 rows, each containing 15 desks, is a problem that can be grasped by most children. The nearer the number in a given concrete multiplicand approaches that of each of several equal groups which can be seen and counted by the pupils, the more useful is the problem for a beginning recitation in multiplication. If the rows in the class-

$$\begin{array}{r} 15\ \text{d.} \\ \times\ \ 2\phantom{\ \text{d.}} \\ \hline 30\ \text{d.}\ \textit{Ans.} \end{array}$$

room contain any number of desks greater than 10, this number may be taken.

It is not wise to employ splints, beans, or the like, as their use above the lowest grade is entirely unnecessary.

Problems involving cents, or dollars and cents, should begin, of course, with those in which the multiplicand contains the cents, such as the cost of 3 articles at 28 cents each. The cost of 20 oranges at 3 cents each may be used in an oral problem to show that even when 28 times 3 cents is written in the accompanying form, the result is obtained by using 3 as the multiplier.

$$\begin{array}{r} 3\ \text{¢} \\ \times\ 28\phantom{\ \text{¢}} \\ \hline 84\ \text{¢} \end{array}$$

CHAPTER V

DIVISION

Division by 2 may be taken up after multiplication by 2 is completed, or it may be postponed for a few months. The connection between multiplication and division is so close that it is probably just as well to take advantage of it by going directly from multiplication by 2 to division by 2.

It is begun, as are all other processes, by oral problems, which should, at first, be such as involve division, rather than "partition," so called.

The following are types:

(*a*) When apples are 2 cents each, how many can be bought for 10 cents?

(*b*) At 2 stalks of corn to a hill, how many hills will contain 14 stalks?

The intelligent teacher will, at this time, carefully refrain from any allusion to "concrete divisors," "abstract quotients," etc. The design should be, for the present, to lead pupils to see the need of the new process and to give a preliminary drill in quotients to 9 obtained by using 2 as a divisor. There should be no attempt as yet to discuss the connection between multiplication and division, as the pupils will discover for themselves all they require to know at this stage.

If the teacher believes that an analysis should be given, he may ask successive pupils how each obtained the result. Some may have counted by 2's to 10 in the first example, others may have found the answer by recalling the multiplication of 2's or 5's. The accepted analysis should be given in some such form as this:

(*a*) He can buy as many apples as there are 2's in 10.
(*b*) There are as many hills as there are 2's in 14.

The next step is the drill in such oral examples as the following:

(*a*) How many 2's in 16?
(*b*) How many 2's in 12?
(*c*) How many 2's in 18? etc., etc.

Written work may begin with examples done on the blackboard. In these each figure of the dividend is a multiple of the divisor; as, for instance:

$2\overline{)84}$ $2\overline{)68}$ $2\overline{)246}$ $2\overline{)680}$

$3\overline{)66}$ $3\overline{)69}$ $3\overline{)396}$ $3\overline{)960}$

$4\overline{)84}$ $4\overline{)48}$ $4\overline{)484}$ $4\overline{)840}$

$5\overline{)55}$ $6\overline{)66}$ $7\overline{)707}$ $8\overline{)880}$

Since quotients are now generally written *above* the dividend in long division, some teachers follow the same plan in short division examples; but as answers hitherto have occupied the last line of the written work, it is probably less confusing to continue this practice until the exigencies of long division in the fourth grade call for the necessary change.

The next step is to teach the pupils to use two figures as partial dividends when necessary, and to locate the first figure of the quotient in the proper place.

The following examples are intended for these purposes:

$$2)\underline{128} \qquad 2)\underline{106} \qquad 2)\underline{1468} \qquad 2)\underline{1684} \qquad 2)\underline{1848}$$

In writing answers to the foregoing, the average pupil will naturally divide 12, 10, 14, 16, and 18 by 2 unless he is taught to say "2 into 1 does not go." The teacher who requires this, frequently confuses the beginner by requiring him to say also, "2 into 1 hundred (or 1 thousand) does not go," and to announce the reduction of 1 hundred to 10 tens, its combination with 2 tens to make 12 tens, etc.

After the pupils have worked many examples of the foregoing type, and have learned to locate the first figure of the quotient, later exercises may be led up to by sending a pupil to the blackboard to multiply a number by 2, and then to divide the product by 2. In the division portion he should be led to say aloud, "2 into 13, 6 times and 1 over; 2 into 15, 7 times and 1 over; 2 into 16, 8 times; answer, 678."

$$\begin{array}{r} 678\\ \times 2\\ 2)\overline{\underline{1356}} \end{array}$$

$$\begin{array}{r} {\scriptstyle 111}\\ 2)1356\\ \hline 678\ \textit{Ans.} \end{array}$$

The writing of a partial remainder in connection with the succeeding figure of the dividend, as shown in the accompanying example, should not be permitted.

Division exercises follow the order suggested under multiplication. After division by 2 is completed, the teacher should give examples in which the divisors extend to 9, but have quotients composed exclusively of 0, 1, and 2. These are prepared in advance by the teacher, who multiplies 210, 221, 121, 212, etc., by 9, 8, etc., using the product as the dividend, and the multiplier as the divisor. The following are types:

$$5)\underline{1050} \qquad 6)\underline{1326} \qquad 7)\underline{847} \qquad 8)\underline{1696}$$

$$9)\underline{1890} \qquad 8)\underline{1768} \qquad 6)\underline{726} \qquad 7)\underline{1484}$$

Pupils should be required to test each division example by multiplying the quotient by the divisor, which gives the dividend when the latter is a multiple of the quotient and the work is correctly done. The dividend should be covered until the product is obtained.

2)1974
987 *Ans.*
×2
1974

The early examples should have exact quotients. When remainders occur in the later examples, they should be written above the divisor in the form of a fraction. The pupil should announce his answer to the foregoing as 256 and 3 fifths, unless the teacher prefers 256, remainder 3. In testing the foregoing result, 3 should be added at once to 5 times 6.

5)1283
$256\frac{3}{5}$ *Ans.*
5
Test, 1283

As a preliminary to long division, sight examples in dividing by 10 are given, followed by written exercises having divisors ending in 0, and worked by short division. Some teachers prefer to separate the terminal ciphers from the preceding figures by a vertical line instead of canceling them. This is possibly the better way, especially as it is the more useful form in examples in which there is a remainder, which will necessarily be the case when the dividend does not end in a cipher. In an example of this kind, the first step after setting off the cipher in the divisor and the last figure in the dividend is to write the latter above the divisor in the form of a fraction. In the given example, 85 ÷ 3 gives 28 as the quotient with 1 as the remainder. This 1 is then written before the 6, making 16 the complete remainder. When remainders are written in the form of a fraction, the latter is not reduced to lowest terms, the answer to 850 ÷ 30, for instance, being given as $28\frac{18}{30}$.

$3\cancel{0})84\cancel{0}$
28 *Ans.*

3|0)84|0
28 *Ans.*

3|0)85|6
$\frac{6}{30}$

3|0)85|6
$28\frac{16}{30}$ *Ans.*

The first long division example should have 21 for the divisor, and 11 for the quotient. A pupil is sent to the board and the example is dictated. If he draws a line below 231, the teacher tells him to place it above and to write the quotient on the line. The boy says 21 into 23 once and 2 over. He is told not to bother with the remainder now, merely to write the first quotient figure in the proper place, which he can generally do correctly. He is then told to write the product of the divisor, 21, by the quotient figure 1, under the proper figures of the dividend. The remaining steps are performed under direction and the answer obtained is tested.

$$\begin{array}{r} \overline{11}\ \textit{Ans.} \\ 21)231 \\ \underline{21} \\ 21 \\ \underline{21} \end{array}$$

Another pupil may be sent to the board to divide 462 by 21, a third to divide 273 by 21, etc. The teacher should prepare a number of examples in which 21 is the divisor, and having exact quotients, which should contain only 1's, 2's, 3's, and 4's at first. The next divisor should be 31, with quotients containing only 1's, 2's, and 3's.

Each result should be tested by the pupil when it is obtained. When the quotient is 11 or 12, the divisor and the quotient need not be written again in making the test. The dividend is covered, and the product is written in a single line.

After many such examples are worked at the desks, a pupil is sent to the blackboard to divide, say, 1365 by 21. If he cannot readily determine the first quotient figure, he should be led to see that 136 ÷ 21 is nearly the same as 130 ÷ 20, or 13 ÷ 2, the quotient of which is 6, etc.

$$\begin{array}{r} \underline{65}\ \textit{Ans.} \\ 21)1365 \\ \underline{126} \\ 105 \\ \underline{105} \\ \text{TEST} \\ 65 \\ \underline{130} \\ 1365 \end{array}$$

By this time he should be able to locate the 6 in its proper place, etc. The advantage of having early two-figure divisors terminate in 1 is that it enables the pupil to obtain the approximate quotient figure by dividing the first figure or two figures of the dividend by the first figure of the divisor. Such divisors as 13, 14, 15, etc., to 19, should not be used until near the end of the term.

The method of testing a quotient greater than 12 is shown in the last example, in which the two-figure divisor is made the multiplier. This is done to obtain new partial products by 1 and 2, in order to avoid the possible repetition of a mistake that may have been made in multiplying by 6 and 5 in performing the divisions. The quotient, 65, is considered the multiplicand, and only the partial products are written.

The early dividends in long division should not extend beyond numbers of four figures even though five figures are allowed in the products obtained in the multiplication examples. Long division is not easy for beginners, and the successive difficulties should be introduced gradually. The teacher should prepare the examples in advance, as those in the text-books are apt to increase in difficulty too rapidly.

The young teacher need not hesitate to use divisors of several figures provided he uses proper ones, and limits the dividend in each case to a number of four figures. The trouble in a division example is determined as much by the quotient as by the dividend. The easiest divisor of 4897 is 4897, the quotient being 1.

When the teacher begins with three-figure divisors he should limit himself, at first, to such as 201, 301, 401, etc., which present even fewer difficulties than the average two-figure number. The fact that 2 hundred and something is contained three times in 6 hundred and something is more apparent than that the quotient of sixty by twenty is 3. In the former the 2 of the divisor is emphasized in a way that is not the case with the 2 in 20, especially when both are said aloud. For the same reason there can be no valid objection to the employment of such 4-figure divisors as 2003, 4006, 3001, etc., when the dividends are limited to numbers of four figures and the quotients are exact, especially as the quotient is limited thereby to a number of one figure.

An example or two in which the quotient contains a cipher should be worked on the blackboard by the pupils.

```
     2
41)8323
```

```
   203 4/41 Ans.
41)8327
   82
    127
    123
      4
  TEST
   207
  812
  8327
```

A pupil called upon to divide 8323 by 41 should be asked, after he has written the first quotient figure, 2, above the 3 of the dividend, how many figures the quotient will contain, and he should be able to say that it will contain three figures. He then works the example. Another pupil, sent to the board to divide 8327 by 41, should also be asked to tell how many figures there will be in the quotient. He then works the example, writing the remainder as a fraction. It will not be necessary to call the attention of the pupils to the fact that $203\frac{4}{27}$ is considered a number of three figures, unless some pupil makes inquiry.

In this test the remainder is combined at once with the first partial product, making the result 207.

The division of 9430 and of 9446 by 41 gives in each case a quotient ending in a cipher.

After sufficient practice with the simpler divisors, such as, 13, 23, 22, 209, etc., etc., work with the more difficult ones might as well commence with 99 as with any other. The quotient figures of 5700 ÷ 99 are the first two figures of the dividend; the only difficulty for the beginner consists in obtaining the partial products. The foregoing example is a sight one in an upper grade; but at this stage it is used to teach pupils to take as a trial divisor a number larger than the original divisor.

In dividing 5721 by 99, the pupil, using 100 as a trial divisor, finds that it is contained 5 times in 572, the first partial dividend, and 7 times in the second partial dividend, 771.

$$\begin{array}{r} 57\tfrac{78}{99}\ \textit{Ans.} \\ 99\overline{)5721} \\ \underline{495} \\ 771 \\ \underline{693} \\ 78 \end{array}$$

$$\begin{array}{r} 5 \\ 99\overline{)5742} \\ \underline{495} \\ 792 \end{array}$$

The use of 99 as a divisor should be extended to examples such as 5742 ÷ 99, in which the pupil trying 7 as the second figure of the quotient obtains a remainder of 99, which shows him that 8 is the figure required. The pupil's first attempt to obtain a partial quotient should bring him within at least 1 of the correct figure. The error, if any, made in dividing by 99, should always consist in taking a figure too small by 1; in the foregoing example, 7 instead of 8.

$$\begin{array}{r} 80\tfrac{8}{99} \\ 99\overline{)7928} \end{array}$$

$$\begin{array}{r} 45 \\ 19\overline{)8664} \\ \underline{76} \\ 106 \\ \underline{95} \\ 114 \\ \underline{114} \end{array}$$

The next divisor might be 19, the pupil using 20 as a trial divisor. The first quotient figure is obtained by thinking 20 into 86, or 2 into 8; the second, by 2 into 10; the third, by 2 into 11. If the pupil tries 5 as the last figure of the quotient, he will find that he has made a mistake. By working many examples he observes that combinations like 114 ÷ 19 may give a result more than 5, etc., and he tries 6, etc., mentally before settling upon the figure he writes as the partial quotient. The class as a whole should not be permitted to make side calculations to determine the quotient figures.

Other divisors ending in 9 may then follow: 29, 39, 49, etc., for instance, but not necessarily in the foregoing order. As a preliminary to the use of such divisors, German teachers employ sight drills, of which the following are types:

(*a*) $30\overline{)180}$ $40\overline{)320}$ $60\overline{)240}$ $90\overline{)630}$

(*b*) $29\overline{)180}$ $39\overline{)320}$ $59\overline{)240}$ $89\overline{)630}$

(*c*) $31\overline{)180}$ $41\overline{)320}$ $61\overline{)240}$ $91\overline{)630}$

Each of these sets contains a large number of examples. When these are used on the blackboard, the teacher places a pointer on one of the combinations in set (*a*), and requires a prompt answer from the pupil called upon. In taking up set (*b*), the teacher leads the pupils to think of the quotient of 180 ÷ 30 when that of 180 ÷ 29 is asked for, and to say 6, omitting the remainder. In giving answers to the drills in set (*c*), the pupil is expected to give the result of 180 ÷ 31, as 5 (omitting the remainder), his attention being directed to the fact that as 180 contains 30 just 6 times, it contains a number larger than 30 fewer than 6 times.

When a divisor is to be used in many examples, special drills are sometimes employed, those for 19 containing the following types:

(*a*) $19\overline{)120}$ $19\overline{)160}$ $19\overline{)100}$ $19\overline{)80}$ $19\overline{)140}$

(*b*) $19\overline{)130}$ $19\overline{)170}$ $19\overline{)110}$ $19\overline{)90}$ $19\overline{)150}$

(*c*) $19\overline{)135}$ $19\overline{)175}$ $19\overline{)115}$ $19\overline{)95}$ $19\overline{)155}$

etc. etc. etc. etc. etc.

Each of these sets is extended to include seven or eight combinations, as is indicated in sets shown on the next page.

In giving answers to drills in (*a*), the pupil mentally substitutes 20 for 19 and replies quickly. In (*b*) and (*c*) he obtains an approximate result by using 20 as the divisor, and he tests the next higher number. Thus, before giving the answer to 19)170, he obtains 8 as the result of 17 ÷ 2, and then multiplies 19 mentally by 9, the product of which he finds to be 171, which is greater than the dividend; he therefore answers "8." In the early stages of a drill in types (*b*) and (*c*) the pupils should be encouraged by the allowance of a liberal time to obtain and state each quotient, no remainder being required.

In later drills, for all of the dividends the divisor need appear only once. The latter should be placed on the blackboard in order that the pupils will have its digits before them. The drills for 49 might be the following:

(*a*)	300	450	150	400	250	350	200
(*b*)	310	460	160	410	260	360	210
(*c*)	290	440	140	390	240	340	190
(*d*)	305	455	155	405	255	355	205
(*e*)	295	445	145	395	245	345	195
		etc.		etc.		etc.	

÷ 49

These dividends serve also for such divisors as 48 and 47; or, at a later stage, for any divisor.

It will be observed that (*a*) consists of multiples of 50; that each number in (*b*) is greater by 10 than the corresponding number in (*a*), and that each of those in (*c*) is less by 10; also that those in (*d*) and (*e*) are, respectively, 5 greater and 5 less than the corresponding numbers in (*a*).

When the class conditions afford the opportunity, such drills should be employed. The writing of them on the blackboard does not take much time, and the answers can be given in a few minutes.

Even when the same divisor is not to be used throughout the written work of the day, the foregoing numbers may be used with any divisor, or cards may be prepared, each of which contains a dividend with its divisor. Other forms will suggest themselves to the ingenious teacher.

The regular long division of the second half of the fourth year may be extended gradually to dividends of six figures, the divisors consisting chiefly of two and of three figures. Carefully prepared examples containing divisors of four or five figures may be used to give variety to the work, especially when these examples are made easier than those having divisors of fewer figures. Drills in examples having quotients containing ciphers must not be omitted, since pupils are prone to overlook the cipher, especially when it is the last figure, as in the accompanying example.

```
     120 170/219 Ans.
219)26450
    219
    ---
     455
     438
     ---
     170
```

In testing this result the pupil should take 120 as the multiplier, and he should write the product of 219 by 120 in one line. In a higher grade, 170 is "added in" at the time of the multiplication, one line being sufficient for the completed result, instead of the three shown above.

```
 TEST
26280
  170
-----
26450
```

Since pupils naturally employ the more convenient number as the multiplier in testing a quotient, the teacher should require the use of the divisor for this purpose when the latter contains no more figures than the quotient. The incorrect result in the following example shows the reason for the rule:

```
     WRONG           DEFECTIVE        CORRECT
     ANSWER            TEST            FORM

       357 18/246     (a) 246         (b) 357
    246)87830              357             246
        738                ---             ---
        ---               1712 *          2142
        1403             1230            1428
        1230             738             714
        ----             -----           -----
         1730            87812           87822
         1712 *             18              18
         ----            -----           -----
           18            87830           87840
```

In (*a*), in which the divisor is multiplied by the quotient, the partial products are the same as those of the division example, so that a mistake in the latter, due to imperfect acquaintance with the table of 7's, is likely to be repeated when 7 is again used to multiply 246, as is shown above. When 246 is the multiplier, as in (*b*), this particular error is eliminated. Later on, other forms of testing results will be given, which will make this rule unnecessary; but at this stage it gives practice in multiplication, which is not undesirable.

As was stated at the beginning of this chapter, the introduction to division by 2, 3, etc., is best made by the use of problems of the "division" type. At any subsequent recitation, oral "partition" problems may be given with the others. The difference in the analysis of the two types, which is insisted upon by many teachers, should not be required of young children.

In the following examples, (*a*) is one in division and (*b*) is one in partition.

(*a*) When apples are 2 cents each, how many can you get for 8 cents?

(*b*) If 8 marbles are divided between 2 boys, William and John, how many does each boy get?

A first-year pupil, called upon to determine the answer to each by using counters, would pick up 2 at a time, arranging them possibly in sets of two, thus:

00 00 00 00

then counting the number of sets, to ascertain the number of apples.

To determine the number of marbles, the child would again pick up 2 at a time, but he might arrange them differently, placing one counter for John and another for William underneath John's. He would then pick up 2 more and place one in

0 0 0 0
0 0 0 0

the first row for John, and the other in William's row. When he had finished he would find 4 counters in each row.

This different arrangement of the counters is supposed to indicate the need of a different form of analysis for each example, the proper one for the second being assumed to state that each boy would get $\frac{1}{2}$ of 8 marbles.

Inasmuch as the child in each case picks up 2 at a time, there is no special reason to believe that he is aided in the solution of written problems by being forced to use two different forms of analysis to show that he obtains 4 by dividing 8 by 2.

The analysis of (*b*), when one is required, should be, "Each boy will get as many marbles as there are 2's in 8."

At this period stress should be laid on division, and $\frac{1}{2}$ of 8 is an example in multiplication of fractions, as the pupil will note later on, when he multiplies 18 by $1\frac{1}{2}$. It is also a mistake to accept in an early analysis any numbers but those given in the problem. The form that says "Each boy will receive $\frac{1}{2}$ of 8 marbles" is incomplete unless it adds, or prefixes, "since 1 boy is $\frac{1}{2}$ of 2 boys."

Some believers in the advisability of making clear to young children the meaning of "partition" think the full and complete analysis outlined above should be required, thereby transforming a simple division example into a compound one, involving both ratio and fractions.

It is difficult to employ such a form of analysis in the solution of this fifth-year problem:

"Find the cost of a pie when $1\frac{1}{2}$ pies cost 15 cents."

When the pupil in a higher grade meets one of this type, he almost instinctively thinks of dividing 15 by $1\frac{1}{2}$, instead of trying to reduce the complex fraction having 1 for its numerator and $1\frac{1}{2}$ for its denominator to the equivalent simple one, $\frac{2}{3}$; and although he finally multiplies 15 by $\frac{2}{3}$, he indicates the required operation as $15 \div 1\frac{1}{2}$.

When division is postponed until the third year, the question of counters does not enter, as it might when all combinations of numbers to 9 are taught in the first year.

The earlier division drills will correspond with the multiplication drills in the thirty-six combinations, and these should be reviewed occasionally throughout the fourth year.

The other oral multiplication drills suggested for the second half of the third grade and for each half of the fourth grade will indicate the nature of the required division drills.

No formal division tables need be made or memorized. Except in the case of the special long division drills, in which the remainders are ignored, dividends in the combinations should be limited to multiples of the divisor. The object of this limitation is to strengthen the pupil's acquaintance with multiplication in each of the earlier drills; such as:

$$56 \div 8, \qquad 42 \div 6, \qquad 63 \div 7, \text{ etc.}$$

giving one of the thirty-six primary products and a factor from 2 to 9, from which to determine the other. It is considered that drills in ascertaining remainders have too small a value to make it advisable to take time from the more useful ones. When a pupil knows the primary products, he can generally calculate the remainders.

The form $90 \div 3$, $80 \div 20$, may be used in the later drills, the one suggested for the early lessons, $3\underline{)18}$, being employed at that time to present the numbers in the order they have in the written work. The pupil may be advised to say "3 into 18" when he has occasion to read the expression 3)18, while $90 \div 30$ should be read "90 divided by 30." As the first is used for perhaps a year before the second appears, the necessity for confining the pupils to a single form no longer obtains. Writing the divisor beneath the dividend, as a fraction, is a third form. Although pupils become familiar with it through the writing of remainders in this way, they do not recognize it as a mark of division. In order to make unnecessary the rule for the reduction of an improper fraction to a whole or a mixed number, many teachers in some of the later drills present division examples in this form:

$$\frac{80}{4} \qquad \frac{60}{20} \qquad \frac{90}{3} \qquad \frac{200}{50} \text{ etc.}$$

These are read "80 divided by 4, 60 divided by 20," etc. The expression "60 over 20," which is objected to by the purists, should not be employed in these grades, since the only object of the form at this time is to lead up to the knowledge of the fact that a fraction is an indicated division, and this is not brought out by the word "over."

The first written problems in division of United States money should be of the partition variety; such as:

When 6 pounds of butter cost \$1.26, what is the price of 1 pound?

The pupil at the board readily sees the answer when the figures are before him. The location of the decimal point is done almost automatically.

$$\begin{array}{r} 6\overline{)\$1.26} \\ \$\ .21\ \textit{Ans.} \end{array}$$

When the pupils reach the other type: How many pounds of sugar at 6 cents a pound can be bought for \$1.26? the teacher should require the problem to be written twice, the terms to be first written with the dollar sign, and then as cents. The pupils have already had problems in which the divisor and the dividend consisted exclusively of cents, and the new example presents no difficulty when changed as shown above.

$$\$.06\overline{)\$1.26}$$

$$\begin{array}{r} 6 \text{ cents}\overline{)126 \text{ cents}} \\ \textit{Ans.}\ \overline{21}\ \text{(lb.)} \end{array}$$

The denomination of the answer is a source of unnecessary worry to teachers, who harass their pupils also with "abstract quotients" and the like. The average child gives himself no concern about it unless he is told to do so by the teacher. When the divisor is abstract and the dividend concrete, the child naturally writes the quotient correctly; in the case of concrete divisor and dividend, the pupil should be permitted to write the denomination of the required result in a parenthesis, and not be required to take an unnecessary line to obtain it, thus: 1 ℔. × 21 = 21 ℔. *Ans.*

CHAPTER VI

FRACTIONS

ORAL examples containing fractions may be commenced as early as the second half of the first year by the question:

(*a*) How many halves are there in a pie?

In the subtraction work, one of the problems may be:

(*b*) If you had a pie and gave one-half of it to your sister, how much of it would you have left?

These problems should be given only when the pupils know what is meant by a half, without special instruction.

At an opportune time during the second year, the pupils' acquaintance with a quarter might be ascertained through inquiries suited to the age, capacity, and environment of the class. These questions should determine what the different members know of the number of quarters in a pie, a dollar, etc.; the number of quarters in half a pie, a half dollar, etc. Paper folding, drawing on the board, and the like, may be employed to make the knowledge definite.

The problems of the second year may, in the first term, include such as involve the following fractions and operations, but should contain nothing more difficult:

(*a*) $\frac{1}{2} + \frac{1}{2}$ (*b*) $1 - \frac{1}{2}$ (*c*) $\frac{1}{4} + \frac{1}{4}$ (*d*) $1 - \frac{1}{4}$

During the second term the work may be extended to include problems involving conditions indicated by the following combinations:

(*a*) $\frac{1}{2} + \frac{1}{4}$ (*b*) $\frac{1}{2} - \frac{1}{4}$ (*c*) $1 - \frac{3}{4}$
(*d*) $\frac{3}{4} - \frac{1}{4}$ (*e*) $\frac{3}{4} - \frac{1}{2}$ (*f*) $2 - \frac{1}{2}$

In the first term of the third year, the meaning of one-third is developed, and the expressions one-fourth and three-fourths are substituted for one-quarter and three-quarters, if this has not already been done. The new oral problems of this grade include the following combinations:

$$(a)\ \tfrac{1}{3}+\tfrac{1}{3} \qquad (b)\ 1-\tfrac{1}{3} \qquad (c)\ 1-\tfrac{2}{3} \qquad (d)\ \tfrac{2}{3}-\tfrac{1}{3}$$

the other fractions being extended to the following:

$$\begin{array}{lll} (a)\ 1\tfrac{1}{2}+\tfrac{1}{2} & (b)\ 1\tfrac{1}{2}+\tfrac{1}{4} & (c)\ 1\tfrac{3}{4}+\tfrac{1}{4} \\ (d)\ 1\tfrac{3}{4}-\tfrac{1}{2} & (e)\ 1\tfrac{1}{2}-\tfrac{1}{4} & (f)\ 1\tfrac{3}{4}-\tfrac{1}{4} \end{array}$$

This work should be altogether incidental and occasional, and should by no means interfere with the regular work of the grades in whole numbers.

During the second half of the third year the written addition of two or three mixed numbers is taken up, which should be preceded by such sight examples as:

$$\begin{array}{rrrrrr} (a)\ 1\tfrac{1}{2} & (b)\ 2\tfrac{1}{2} & (c)\ 1\tfrac{1}{4} & (d)\ 2\tfrac{3}{4} & (e)\ 2\tfrac{1}{3} & (f)\ 2\tfrac{2}{3} \\ \underline{+\ 1\tfrac{1}{2}} & \underline{+\ 1\tfrac{1}{2}} & \underline{+\ 1\tfrac{1}{4}} & \underline{+\ \ \tfrac{1}{4}} & \underline{+\ 1\tfrac{1}{3}} & \underline{+\ 1\tfrac{1}{3}} \end{array}$$

the fractions in each example being either halves alone, fourths alone, or thirds alone.

Sight examples in subtraction embrace the following, the minuend in each being either a mixed number containing a fraction at least as large as that in the subtrahend, or a whole number:

$$\begin{array}{rrrrr} (a)\ 6\tfrac{1}{2} & (b)\ 5\tfrac{3}{4} & (c)\ 7\tfrac{2}{3} & (d)\ 9\tfrac{1}{4} & (e)\ 8\tfrac{1}{3} \\ \underline{-\ 2\tfrac{1}{2}} & \underline{-\ 1\tfrac{3}{4}} & \underline{-\ 1\tfrac{2}{3}} & \underline{-\ 6\tfrac{1}{4}} & \underline{-\ 3\tfrac{1}{3}} \\ \\ (f)\ 7\tfrac{2}{3} & (g)\ 6\tfrac{3}{4} & (h)\ 8\tfrac{2}{3} & (i)\ 5\tfrac{3}{4} & (j)\ 9\tfrac{3}{4} \\ \underline{-\ 7\tfrac{1}{3}} & \underline{-\ 6\tfrac{1}{4}} & \underline{-\ 1\tfrac{1}{3}} & \underline{-\ 2\tfrac{1}{4}} & \underline{-\ 9\tfrac{1}{2}} \\ \\ (k)\ 6 & (l)\ 7 & (m)\ 5 & (n)\ 3 & (o)\ 4 \\ \underline{-\ 5\tfrac{1}{2}} & \underline{-\ 6\tfrac{1}{3}} & \underline{-\ 4\tfrac{1}{4}} & \underline{-\ 2\tfrac{3}{4}} & \underline{-\ 3\tfrac{2}{3}} \end{array}$$

$$\begin{array}{rrrrr} (p)\ 9 & (q)\ 8 & (r)\ 7 & (s)\ 8 & (t)\ 9 \\ -1\frac{1}{2} & -1\frac{1}{3} & -1\frac{1}{4} & -1\frac{2}{3} & -1\frac{3}{4} \\ \hline \end{array}$$

$$\begin{array}{rrrrr} (u)\ 7 & (v)\ 6 & (w)\ 5 & (x)\ 4 & (y)\ 3 \\ -3\frac{1}{2} & -2\frac{1}{4} & -3\frac{1}{3} & -2\frac{2}{3} & -1\frac{3}{4} \\ \hline \end{array}$$

It is not supposed that all of these exercises, either in addition or in subtraction, can be handled successfully at one recitation. The teacher should limit the first to exercises in which the fractions are halves alone, one or more to those in which the fractions are fourths alone, and one or more to those in which the fractions are thirds alone, the suggested examples being used for review purposes.

The first written exercises should contain but three addends, not more than two of them being mixed numbers. All but one of the numbers might well be 10 or less; the object being to work as many examples as possible in addition of fractions, and not to provide a drill in addition of whole numbers. It is expected that in these exercises the pupils will write nothing but the answers, for which reason the sum of the fractions in each should not exceed 1.

As a preliminary to exercises in which the sum of the fractions is greater than 1, the sum of three addends is used as a sight exercise, each addend containing $\frac{1}{2}$. The pupils should be permitted in this to combine two of the halves into 1, without being required to obtain $\frac{3}{2}$ as the sum of the fractions and then to reduce the improper fraction to $1\frac{1}{2}$. Other sight examples containing three addends may also be given, in each of which the sum of two of the fractions is 1. The need of adding the numerators may be shown by the following exercises:

$$\begin{array}{r} 2\frac{1}{2} \\ \frac{1}{2} \\ 3\frac{1}{2} \\ \hline \end{array}$$

$$\begin{array}{r} \frac{2}{3} \\ 1\frac{1}{3} \\ 2\frac{1}{3} \\ \hline \end{array}$$

$$\begin{array}{r} 1\frac{1}{4} \\ 2\frac{3}{4} \\ 3\frac{1}{4} \\ \hline \end{array}$$

$$\begin{array}{r} \text{2 thirds} \\ \underline{+\ \text{2 thirds}} \end{array} \qquad \begin{array}{r} \frac{2}{3} \\ \underline{+\ \frac{2}{3}} \end{array} \qquad \begin{array}{r} 1\frac{2}{3} \\ \underline{+\ \frac{2}{3}} \end{array} \qquad \begin{array}{r} 2\frac{2}{3} \\ \underline{+\ 1\frac{2}{3}} \end{array}$$

Up to this time the words, numerator, denominator, etc., are not used. If the teacher finds it necessary to speak of "fractions" or of "mixed numbers," he should do so, but no definition is required. A child recognizes a rhinoceros from a picture in his reader or from having previously seen one in a menagerie, and not through any memorized description.

Successive pupils may be sent to the blackboard, each to write the answer under one of the foregoing combinations. The first result may be written "4 thirds." If a pupil writes $\frac{4}{3}$ as the answer to the next, he may be asked to state it as a "mixed number," if this expression has already been used. If he is unable to do so, one of his mates will give the required answer, which the other then writes. Larger numbers may be employed to develop the proper method, which consists of first adding the fractions, then mentally reducing the improper fraction to a mixed number, writing the fractional part in its place, and carrying the whole number to the ones' figure of the integral part of the first addend.

$$(a)\ \begin{array}{r} 1\frac{3}{4} \\ 1\frac{3}{4} \\ \underline{1\frac{3}{4}} \end{array} \qquad (b)\ \begin{array}{r} 1\frac{2}{3} \\ 1\frac{2}{3} \\ \underline{1\frac{2}{3}} \end{array} \qquad (c)\ \begin{array}{r} 2\frac{3}{4} \\ \underline{+2\frac{3}{4}} \end{array}$$

The sum of (*a*) may then be found. In (*b*) the sum of the fractions is a whole number. By means of (*c*) the practice of writing $\frac{1}{2}$ in a result instead of $\frac{2}{4}$ is again brought to mind. This reduction should of course be made mentally and before the fractional part of the answer is written. The time available for this work should be spent almost entirely in abstract exercises, sight and written, comparatively few written problems being presented for solution.

The first term of the fourth year introduces into an addition or a subtraction exercise two fractions having different denominators, one set consisting of halves and fourths, another consisting of halves and sixths, and a third consisting of thirds and sixths. Those containing halves and fourths present no difficulty. When the pupils have developed through circles drawn on the blackboard the relation between $\frac{1}{2}$ and $\frac{1}{6}$ and that between $\frac{1}{3}$ and $\frac{1}{6}$, the subsequent procedure will depend upon the development of the pupils and the time that can be devoted to this particular topic. The children should obtain of themselves the answers to the following combinations, employing circles, if necessary, for the purpose:

(*a*) $\frac{1}{6} + \frac{1}{6}$ (*b*) $\frac{1}{2} + \frac{1}{6}$ (*c*) $\frac{1}{3} + \frac{1}{6}$ (*d*) $\frac{2}{3} + \frac{1}{6}$
(*e*) $\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ (*f*) $\frac{1}{2} + \frac{1}{6} + \frac{1}{6}$ (*g*) $\frac{1}{3} + \frac{1}{6} + \frac{1}{6}$

The use of circles rather than of rectangles is recommended, as their divisions are more likely to suggest parts of things.

Just as soon as the class can obtain the results understandingly without these circles, the latter should be dispensed with; pupils still requiring their aid being permitted, however, to continue their use.

The recommendation that pupils of the lower grades should not be bothered with ones, tens, etc., does not apply here. These older children are ready to understand the fact that only like numbers can be combined, even if they are not supposed to make the statement. The addition of $\frac{1}{2}$ and $\frac{1}{4}$ may not as yet have brought into consciousness the fact that they have changed $\frac{1}{2}$ to $\frac{2}{4}$ in order that the fractions should have the same denominator before they are combined. This may be called to their attention by placing on the blackboard the following, and getting a pupil to say that he changed 1 half to 2 fourths, which he may write as

(*a*) 1 half
\+ 1 fourth
____

shown at (*b*), the sum being written with the denominator written as "fourths." The word "denominator" is used by the teacher, and possibly the expression "common denominator."

$$\begin{array}{l} (b)\ \text{2 fourths} \\ \underline{+\ \text{1 fourth}} \end{array}$$

If it be deemed necessary, each of the combinations specified above may first be written in the same way by successive pupils, and the answer given in lowest terms, this expression "lowest terms" perhaps not being employed as yet.

One pupil may be permitted to write both forms as shown at (*a*) and (*b*). Just at this point it is better to go too slowly than to make undue haste. Later work in fractions will proceed all the more rapidly after the pupils have obtained a fair understanding of this topic by means of these preliminary lessons.

$$\begin{array}{l} (a)\ \text{1 sixth} \\ \underline{+\ \text{1 sixth}} \\ \quad \text{2 sixths} = \text{1 third} \end{array} \qquad \begin{array}{l} (b)\ \frac{1}{6} \\ \underline{+\ \frac{1}{6}} \\ \quad \frac{1}{3}\ \textit{Ans.} \end{array}$$

When a sufficient number of the combinations have been treated in this way, the teacher may have a pupil write on the blackboard exercises requiring the addition of two or three mixed numbers, the fractions in each set being the same as those of the preceding combinations. These may be used in the several ways suggested in a previous chapter. They should not, however, be employed in oral exercises, it being inadvisable to subject pupils at this stage to the strain involved in combining fractions other than those they are thoroughly familiar with outside of school, such as $\frac{1}{2}$ and $\frac{1}{4}$.

The subtraction exercises of this year continue to have an integral minuend, or a mixed number minuend in which the fraction is at least as great as that in the subtrahend.

Addition and subtraction of fractions are concluded in primary grades by the use of such combinations as:

$$\tfrac{1}{2} + \tfrac{1}{3} \qquad \tfrac{1}{2} + \tfrac{1}{5} \qquad \tfrac{1}{3} + \tfrac{1}{4} \qquad \tfrac{1}{3} + \tfrac{1}{5} \qquad \tfrac{1}{4} + \tfrac{1}{6}$$

or as many of them as can be handled in the period allotted to this topic.

By this time, the pupils should be ready to add the fractions given in the accompanying example, (*a*). Since they have combined $\frac{1}{2} + \frac{1}{6}$ and $\frac{1}{3} + \frac{1}{6}$, the pupil sent to the blackboard should realize that 6 is the common denominator and proceed. If he does not, some member of the class will doubtless give the necessary hint. After (*a*) has been finished, (*b*) is dictated to another pupil, who should determine from looking at (*a*) that 6 is the common denominator. In the work of the preceding term the denominator of one of the fractions to be combined is always a multiple of the other denominator; at this stage, the denominators are prime to each other, but it should be unnecessary to state the fact that their product furnishes the common denominator.

$$(a)\quad \begin{array}{r} 1\frac{1}{2} \\ 2\frac{1}{3} \\ \underline{3\frac{1}{6}} \end{array}$$

$$(b)\quad \begin{array}{r} 1\frac{1}{2} \\ \underline{2\frac{1}{3}} \end{array}$$

After a few examples involving halves and thirds, the pupils are ready for the others, without numerous preliminary exercises.

Up to this point, the pupils should be able to write the results without any side calculations. They should be expected to continue to do so with examples containing only two fractions, provided each has 1 for its numerator. When the fractions contain other numerators, side calculations are admissible.

$$\begin{array}{r} 3\frac{1}{4} \\ 17\frac{1}{5} \\ \underline{6} \\ 26\frac{9}{20} \end{array}\ \textit{Ans.}$$

There are various forms of arranging exercises in addition of fractions. Some teachers add, for instance,

$$3\tfrac{4}{5} + 17\tfrac{3}{4} + 9\tfrac{3}{10}$$

by first finding the sum of the fractions, in some such way as this:

$$\tfrac{4}{5} + \tfrac{3}{4} + \tfrac{3}{10} = \tfrac{16}{20} + \tfrac{15}{20} + \tfrac{6}{20} = \tfrac{37}{20} = 1\tfrac{17}{20}$$

and then combining this result with the integers,

$$3 + 17 + 9 + 1\tfrac{17}{20} = 30\tfrac{17}{20}. \quad \textit{Ans.}$$

A more convenient form consists in writing the common denominator only once, placing it above the new numerators, the latter being separated from the original fractions by a vertical line. The former are then added, their sum is written above the common denominator, and the result, if an improper fraction, reduced to a mixed number. The fractional part of this is written under the fraction in the original example and the integral part is carried.

$$\begin{array}{r|r}
 & \underline{20} \\
3\tfrac{4}{5} & 16 \\
17\tfrac{3}{4} & 15 \\
\underline{9\tfrac{3}{10}} & \underline{6} \\
\textit{Ans. } 30\tfrac{17}{20} & \tfrac{37}{20} = 1\tfrac{17}{20}
\end{array}$$

$$\begin{array}{rcl}
3\tfrac{4}{5} & = & 3\tfrac{16}{20} \\
17\tfrac{3}{4} & = & 17\tfrac{15}{20} \\
\underline{9\tfrac{3}{10}} & \underline{=} & \underline{9\tfrac{6}{20}} \\
 & & 29\tfrac{37}{20} \\
 & & = 29 + 1\tfrac{17}{20} \\
 & & = 30\tfrac{17}{20} \quad \textit{Ans.}
\end{array}$$

A few teachers prefer the accompanying form, which shows the fractions after their reduction, each having its denominator. This arrangement requires the pupil, in adding the fractions, to skip over the denominators, which occasionally causes errors.

Subtraction examples are arranged in the same way as addition examples, side calculations being employed only when necessary. As the subtraction exercises of this grade containing two mixed numbers should require that the fraction in the minuend be greater than the one in the subtrahend, many pupils will be able to obtain the result without writing out the new numerators.

$$\begin{array}{r|r}
 & \underline{15} \\
16\tfrac{4}{5} & 12 \\
\underline{9\tfrac{2}{3}} & \underline{10} \\
\textit{Ans. } 7\tfrac{2}{15} & \tfrac{2}{15}
\end{array}$$

$$\begin{array}{r}
16\tfrac{2}{3} \\
\underline{-\;9\tfrac{4}{5}} \\
6\tfrac{1}{5} + \tfrac{2}{3}
\end{array}$$

If a case comes up in which a pupil needs to subtract $9\tfrac{4}{5}$ from $16\tfrac{2}{3}$, for instance, the teacher could get him to see that he knows how to take $9\tfrac{4}{5}$ from 16, and that to this result $\frac{2}{3}$ should be added. When he appreciates this fact, he will have learned a method that will be useful to him later.

While pupils are solving oral problems involving division by 2, one should occasionally be given involving the finding of one-half of an even number. These should be mingled with others without directing the attention of the pupils to them in any way, as their experience in and out of school has familiarized them with the expression and its significance.

The following types are suggested:

(*a*) Mary had 18 cherries. She gave one-half of them to her brother. How many cherries did she give him?

(*b*) What is the cost of a half of a 10-cent pie?

The drill exercises should include a few examples presented in this form:

$\frac{1}{2}$ of 6 $\frac{1}{2}$ of 16 $\frac{1}{2}$ of 8

The fact that the result is obtained by employing 2 as a divisor should not be brought out. The analysis of (*a*) states merely that she gave him $\frac{1}{2}$ of 18 cherries, or 9 cherries.

As a preliminary to written exercises, a pupil is sent to the board and told to write "one-half of 248," and to place after it the sign of equality. He writes it as here shown, and is then told to write the answer, the work finally appearing in this shape:

$\frac{1}{2}$ of 248 =

$\frac{1}{2}$ of 248 = 124. *Ans.*

The pupil divides 248 by 2, but the attention of the class is not called to the fact. All that should be insisted upon is that the result be obtained without writing any other figures than those given above. This is made possible by limiting the written exercises to numbers of two figures, at first employing only even numbers, as for instance:

$\frac{1}{2}$ of 30 $\frac{1}{2}$ of 36 $\frac{1}{2}$ of 56 $\frac{1}{2}$ of 90

Later on, the sight and the written exercises may include odd numbers; such as, $\frac{1}{2}$ of 9, $\frac{1}{2}$ of 17, etc., $\frac{1}{2}$ of 29, $\frac{1}{2}$ of 51, etc. The result in this case is always written as a mixed number, thus:

$\frac{1}{2}$ of 33 = 16$\frac{1}{2}$. *Ans.*

The oral problems should also include an occasional one requiring the finding of a number when one-half of it is given. The following are typical:

(*a*) When one-half of a pie costs 5 cents, what does the whole pie cost?

(*b*) Mary gave her brother one-half of her cherries. She gave him 9 cherries. How many did she have at first?

These problems should not be analyzed. The pupils are as yet unable to realize that the answers are obtained by dividing 5 by $\frac{1}{2}$, and 9 by $\frac{1}{2}$. They merely multiply by 2, which produces the same result. There should be no written problems of this kind just yet, nor any drill exercises.

A similar procedure should be followed when division by 4 is completed. One-fourth of numbers exactly divisible by 4 is introduced into oral problems. If the pupil's analysis shows that he obtains the cost of one-fourth of a 20-cent pie by first finding $\frac{1}{2}$ of 20 cents, and then $\frac{1}{2}$ of this result, he should be led to say that "$\frac{1}{4}$ of a 20-cent pie will cost $\frac{1}{4}$ of 20 cents, or 5 cents," so that he will realize that but one operation is necessary. The expression "one-fourth," which is here substituted for "a quarter," if not previously done, emphasizes the fact that $\frac{1}{4}$ is 1 of the 4 equal parts, which should, however, not as yet be formulated.

The first drill exercises should be of the same kind, such as

(*a*) $\frac{1}{4}$ of 24 (*b*) $\frac{1}{4}$ of 36 (*c*) $\frac{1}{4}$ of 28

The following are some written types:

(*a*) $\frac{1}{4}$ of 52 (*b*) $\frac{1}{4}$ of 96 (*c*) $\frac{1}{4}$ of 68

The answers to the latter should be written directly, being obtained in the same way as is recommended for the finding of one-half of a number.

The next step includes such examples as:

(*a*) $\frac{1}{4}$ of 53 (*b*) $\frac{1}{4}$ of 99 (*c*) $\frac{1}{4}$ of 69

in which the fractions, $\frac{1}{4}$ and $\frac{3}{4}$, appear in the results. Lastly come those in which the fraction $\frac{2}{4}$ is reduced to $\frac{1}{2}$; such as

(*a*) $\frac{1}{4}$ of 54 (*b*) $\frac{1}{4}$ of 98 (*c*) $\frac{1}{4}$ of 70

In these the pupil should be required to make the reduction before writing the answer.

An occasional oral problem, without comment or analysis should be given, in which the pupils are required to find a number when its one-fourth is given; such as,

(*a*) James gave William one-fourth of his marbles. He gave William 9 marbles. How many did James have at first?

(*b*) When one-fourth of a pound of candy costs 10 cents what is the price of a pound?

Further suggestions as to problems of this type will be given later on.

It must be remembered that the age and the development of primary children preclude a systematic treatment of fractions. These early lessons are introduced chiefly to give the children that will go no farther a working familiarity with the most useful applications. In doing this, a foundation is also laid for the subsequent lessons, a fact which the teacher should bear in mind. In each of these early lessons he should avoid teaching anything in a way that must be radically changed in a higher grade.

The fractions first used are those with which the children are familiar, the half and the quarter. One-third does not come within their previous experiences, so that its introduction is postponed until work with halves and quarters familiarizes them with addition, subtraction, and multiplication of fractions. After this is well done, the pupils are ready to understand the meaning of a third, and to use it in operations. For this reason, one-third of a number does not enter into problem work when division by 3 is completed. One-fifth, one-sixth, etc., should be postponed until the latter half of the fourth year, perhaps, when these fractions are called for in the addition and subtraction work.

The examples that require the finding of $\frac{1}{2}$ of 12, $\frac{1}{4}$ of 8, etc., teach pupils indirectly how to multiply 12 by $\frac{1}{2}$, 8 by $\frac{1}{4}$, etc., an integer by a fraction. Those that are intended to lead up to the multiplication of a fraction by an integer, $\frac{1}{2}$ by 12, for instance, or $\frac{1}{4}$ by 8, are such as the following:

(*a*) How many pints of ice-cream will be required to give 12 girls $\frac{1}{2}$ pint each?

(*b*) Find the cost of 8 baseballs at a quarter of a dollar each.

The teacher should never hesitate to increase the variety of his oral problems by giving any that can be solved by the children without preliminary explanations. The fact that it might be called square root need not deter him from asking: "What number multiplied by itself gives 25?" This does not signify that the teacher should hunt out unusual forms of problems; it means merely that the teacher should not postpone until it is called for by the course of study a problem of any kind that can be solved by the pupils without help. When the type is met in later work the children recognize the new problems as former acquaintances.

The analysis of (*a*) should be to the effect that 12 girls require 12 half-pints, or 6 pints; of (*b*) that 8 baseballs will cost 8 quarter-dollars, or $2.

When the time arrives for two-step operations, the early oral problems should lead up to the use of a mixed number as a multiplier by problems similar to the following:

(*a*) Find the cost of $3\frac{1}{2}$ pounds of sugar at 6 cents a pound.

(*b*) If 20 bushels of wheat can be raised on an acre of land, how many bushels can be raised on $2\frac{1}{4}$ acres?

The first written examples worked on the blackboard should be almost simple enough to be done mentally. If the pupil begins by finding the product by 2, as he has done in the oral problems, the teacher inquires where he generally begins in written addition, multiplication, etc., and tells him to follow the same plan with this example. If a pupil should locate the product by 2 one place to the right, which would make the result 615, he should be asked how many bushels would be yielded by 3 acres, and be led to see that $2\frac{1}{2}$ times 30 is less than 3 times 30, or 90. Questions are put which tend to show that the result must be $\frac{1}{2}$ of 30 added to 2 times 30, and that when he places the product of 30 by 2 one place to the left of the corresponding figure of the number above it, he is writing 60 tens, which is the product of 30 by 2 tens.

$$\begin{array}{r} 30 \\ \times\, 2\frac{1}{2} \\ \hline 15 \\ 60 \\ \hline 75 \end{array}\quad \textit{Ans.}$$

It cannot be too frequently brought to the attention of the pupil that he is expected to avoid serious mistakes by carefully inspecting his answer after performing the calculations. During the operation, he is necessarily required to concentrate his attention upon each successive step. His task is not completed until he has given special attention to the final result.

The following are types of results not infrequently offered by children:

(a)
$$\begin{array}{l} 4)92(113\ \textit{Ans.} \\ \quad\underline{4} \\ \quad 5 \\ \quad\underline{4} \\ \quad 12 \\ \quad\underline{12} \end{array}$$

(b)
$$\begin{array}{l} 21)6384(34\ \textit{Ans.} \\ \quad\underline{63\ \ } \\ \qquad 84 \\ \qquad\underline{84} \end{array}$$

(c)
$$\begin{array}{r} 3\frac{1}{2} \\ \underline{+\,3\frac{1}{2}} \\ 61\ \textit{Ans.} \end{array}$$

Even when no member of a class is guilty of such mistakes in his abstract work, some of them make more egregious blunders in the answers they obtain to written problems.

The mistakes in (*a*) and (*b*) noted above, are prevented by requiring the pupils (1) to write answers above the dividends, (2) to locate the first quotient figure properly, and (3) to arrange the work by keeping numbers in a line, etc. The mistake in (*c*) is largely due to the habit of writing superfluous figures.

$$\begin{array}{r} 195 \\ \underline{\times\ 3\frac{1}{5}} \\ 5)\underline{195} \\ 39 \\ \underline{585} \\ 624\ \ \textit{Ans.} \end{array}$$

The unnecessary work occasionally insisted upon by the teacher of this grade is shown in the accompanying example. He requires that 195 be multiplied by the numerator 1, since the pupils later on will have to multiply by 3, 2, etc., when the fractions are $\frac{3}{4}$, $\frac{2}{3}$, etc.

Other examples are worked at the board in which the multipliers are $2\frac{1}{2}$, $3\frac{1}{4}$, $2\frac{1}{3}$, etc., the multiplicands being multiples of the denominators of the fractions and less than 100. In all of these, the pupil should not be permitted to write any figures beyond those employed in the first, $30 \times 2\frac{1}{2}$, their previous exercises in finding $\frac{1}{2}$ of 96, and the like, having accustomed them to divide by 2 without rewriting the numbers.

$$\begin{array}{r} 27 \\ \times 3\frac{1}{5} \\ \hline 5\frac{2}{5} \\ 81 \\ \hline 86\frac{2}{5} \text{ Ans.} \end{array}$$

$$\begin{array}{r} 88 \\ \times 37\frac{1}{2} \\ \hline 44 \\ 616 \\ 264 \\ \hline 3300 \text{ Ans.} \end{array}$$

After a time multiplicands may be used that are not multiples of the denominator of the fraction in the multiplier. When multipliers of two figures are introduced, a pupil that works the first example may blunder in his location of the partial products by 7 and 3, respectively. In such a case it is often advisable to permit him to proceed to the end. He is then asked to multiply 90 by 40 (using convenient factors) and to compare the last result, 3600, with the one obtained by using two smaller factors, 37½ and 88. A pupil's mistake gives the judicious teacher an opportunity to ask questions that will suggest new methods of testing work. This indirect plan frequently accomplishes more than the direct telling of what should be done.

$$\begin{array}{r} 88 \\ \times 37\frac{1}{2} \\ \hline 44 \\ 616 \\ 264 \\ \hline 32604 \text{ (\textit{wrong})} \end{array}$$

Pupils may test their answers, for the present, by doing the work a second time without looking at the first result. Later, when they have learned two or more ways of working the same example, they will discover methods for themselves.

As products by 1¼, 1½, 1⅓ do not require a line to show the product by 1, this type is postponed for the present. After pupils have learned to employ the general method, the special method may be taught. This requires the product of 176 by 1½ to be indicated by writing the factors on the same line, using 176 as the product by 1, without again writing it. The same form is used with multipliers of two figures when the ones' figure or the tens' figure is 1.

$$\begin{array}{r} 176 \times 1\frac{1}{2} \\ 88\phantom{ \times 1\frac{1}{2}} \\ \hline 264 \text{ Ans.} \end{array}$$

$$\begin{array}{r} 176 \times 21 \\ 352 \\ \hline 3696 \text{ Ans.} \end{array}$$

$$\begin{array}{r} 176 \times 13 \\ 528 \\ \hline 2288 \text{ Ans.} \end{array}$$

Problems involving the multiplication of a number by $1\frac{1}{2}$, $1\frac{1}{4}$, etc., need not be avoided. They can be given as a 2-step problem before pupils learn to multiply by a mixed number. The pupil generally finds the cost of a pound and a quarter of 60-cent candy by writing the cost of $\frac{1}{4}$ pound underneath the price of a pound. If the blackboard work shows superfluous figures, the teacher brings out this fact, through his inquiry for a shorter way that may have been employed by another pupil in working at his seat. A pupil should be expected to go about the solution of a problem in his own way, and he should be permitted to do so, any change he makes in his procedure being due to the selection of a better method from among those seen on the blackboard in solutions furnished by his mates. The one he picks out is chosen because of his ability to understand it. As all children cannot be made to think in the same way, the teacher should see the futility of attempts made to secure uniformity in the methods employed in the solution of problems, oral or written.

$$\begin{array}{rl}60 & \\ \underline{15} & \\ 75 & \textit{Ans.}\end{array}$$

The multiplication of a mixed number by an integer may be developed by such a problem as the following:

(*a*) If a man divides 7 apples equally between 2 boys, how many does each receive?

The next may be of this type:

(*b*) If 2 boys have $4\frac{1}{2}$ apples apiece, how many apples have they together?

After several of this latter type are solved, a boy is asked to work on the board the following:

If 8 boys have $12\frac{1}{2}$ apples apiece, how many have they in all?

The pupil says that the entire number must be 8 times

$12\frac{1}{2}$, which he indicates on the blackboard in the form shown herewith. He is led to say that the entire number is 8 half-apples, or 4 apples, and 8 times 12 apples. He is shown that he need not waste chalk in writing 4, which he may carry to the next product, 8 times 2. Adding 4 to 16, the result is 20, the cipher of which is written and 2 (tens) carried; etc.

$$\begin{array}{lrl} & 12\frac{1}{2} & a. \\ & \times 8 & \\ \hline \textit{Ans.} & 100 & a. \end{array}$$

The subsequent exercises should, for a time, be limited to mixed numbers less than 100 in the multiplicand, with unit numerators to the fractions, the multipliers being numbers less than 13, and multiples, for a time, of the denominators of the fractions in the multiplicand.

Some teachers write these examples in the equation form as blackboard exercises, requiring pupils to write on their papers nothing but the answers, leaving the other arrangement for examples having larger multipliers. Examples similar to the following can be used as blackboard exercises:

(*a*) $26\frac{1}{2} \times 4 =$ etc. (*b*) $33\frac{1}{3} \times 9 =$ etc. (*c*) $37\frac{1}{2} \times 8 =$ etc.

Later examples may have a fraction in the result, and still be worked from the blackboard, such as:

(*a*) $26\frac{1}{2} \times 5 =$ (*b*) $33\frac{1}{3} \times 4 =$ (*c*) $37\frac{1}{2} \times 7 =$

In writing the result in (*a*) the pupil obtains $2\frac{1}{2}$ as the product of 5 times $\frac{1}{2}$. He writes $\frac{1}{2}$, and carries 2, etc.

To prevent pupils from being tempted to use the mixed number as the multiplier when large numbers are introduced, the latter should be simple, such as 22, and the multiplicand should contain more figures and larger ones, as is shown in this example. In this, the product of 22 by $\frac{1}{2}$ should be written, instead of being carried to the next partial product.

$$\begin{array}{rl} 879\frac{1}{2} & \\ \times 22 & \\ \hline 11 & \\ 1758 & \\ 1758 & \\ \hline 19349 & \textit{Ans.} \end{array}$$

The written problems at this time may include the finding of the cost of 4 articles at $37\frac{1}{2}$ cents each, etc.

As a preliminary to the use of $\frac{3}{4}$, $\frac{2}{3}$, etc., as multipliers, a problem may be given asking the cost of three-quarters of a 20-cent pie, or the like. The pupil readily solves it by finding the cost of 1 quarter and multiplying the result by 3. After several such problems, the following sight exercises may be given:

(*a*) $\frac{1}{2}$ of 60 (*b*) $\frac{1}{3}$ of 90 (*c*) $\frac{1}{4}$ of 80 (*d*) $\frac{1}{5}$ of 50
(*e*) $\frac{2}{3}$ of 60 (*f*) $\frac{3}{4}$ of 80 (*g*) $\frac{2}{5}$ of 50 (*h*) $\frac{3}{5}$ of 50
(*i*) $\frac{1}{6}$ of 60 (*j*) $\frac{5}{6}$ of 60 (*k*) $\frac{1}{8}$ of 80 (*l*) $\frac{5}{8}$ of 80
(*m*) $\frac{4}{5}$ of 50 (*n*) $\frac{7}{8}$ of 80 (*o*) $\frac{7}{10}$ of 50 (*p*) $\frac{3}{8}$ of 80

The following examples may be written on the board by a pupil, who later writes the answers as they are announced by different pupils:

(*a*) $\frac{1}{3}$ of 42 = ? (*b*) $\frac{1}{5}$ of 75 = ? (*c*) $\frac{1}{6}$ of 84 = ?
(*d*) $\frac{1}{4}$ of 96 = ? (*e*) $\frac{1}{7}$ of 98 = ? (*f*) $\frac{1}{8}$ of 96 = ?

Underneath those are written the following:

(*a*) $\frac{2}{5}$ of 75 = ? (*b*) $\frac{3}{4}$ of 96 = ? (*c*) $\frac{3}{8}$ of 96 = ?
(*d*) $\frac{2}{3}$ of 42 = ? (*e*) $\frac{5}{6}$ of 84 = ? (*f*) $\frac{4}{5}$ of 75 = ?
(*g*) $\frac{5}{8}$ of 96 = ? (*h*) $\frac{3}{5}$ of 75 = ? (*i*) $\frac{7}{8}$ of 96 = ?

The latter are used as blackboard exercises, the pupils writing only the answers. They obtain $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc., of the numbers from the answers to the previous exercises, which remain on the board.

$$\frac{2}{\cancel{5}} \text{ of } \overset{19}{\cancel{95}} = 38 \textit{ Ans.}$$

Written exercises should at first be limited to such combinations as give answers in whole numbers. An example such as is here given is written on the blackboard from dictation. If the pupil obtains the result by side calculations, the teacher tells him that he can write $\frac{1}{5}$ of 95 above the latter, and draw a line through 5 and 95, to show that they are no longer needed. Having found that 19 is one-fifth of ninety-five, he multiplies it by 2, which gives him the result required; that is, he finds the product of the two remaining numbers.

When the time arrives for the introduction of multiplicands that are not exactly divisible by the denominator of the fraction in the multiplier, the following example may be worked on the board. If the pupil endeavors to employ the same method for this example, he should be permitted to do so, and even helped by the teacher to complete the work.

$$\frac{2}{\cancel{5}} \text{ of } \overset{5\frac{2}{5}}{\cancel{27}} = 10\tfrac{4}{5}$$

He is then informed that another plan is followed when the second number does not exactly contain the first; that instead of dividing 27 by 5 and then multiplying the quotient by 2, 27 is first multiplied by 2, and the product divided by 5. The pupil, observing that the latter result is the same as the first one, is ready to accept the proposed method.

$$\tfrac{2}{5} \text{ of } 27 = \tfrac{54}{5} = 10\tfrac{4}{5} \textit{ Ans.}$$

After a few written exercises of this kind the pupil is ready to multiply 27 by $3\frac{2}{5}$. With the foregoing example on the blackboard, being done once more if it has already been erased, the pupil is ready to understand the arrangement suggested, which places 54, the product by 2, underneath the multiplier, then the quotient of 54 by 5, to which is added 3 times 27, or 81.

$$\begin{array}{r}
27 \\
\times\ 3\frac{2}{5} \\
\hline
54 \\
\hline
10\frac{4}{5} \\
81 \\
\hline
91\frac{4}{5} \textit{ Ans.}
\end{array}$$

In multiplying 26 by $4\frac{3}{4}$ the fraction $\frac{2}{4}$ should be changed to $\frac{1}{2}$ either at the time of writing the result or immediately after, enclosing $\frac{2}{4}$ in a parenthesis.

$$\begin{array}{r}
26 \\
\times\ 4\frac{3}{4} \\
\hline
78 \\
\hline
19\left(\frac{2}{4}\right)\frac{1}{2} \\
104 \\
\hline
123\frac{1}{2} \textit{ Ans.}
\end{array}$$

The remaining type, the product of a mixed number containing any fraction by an integer, follows the same general method. Taking the example $879\frac{3}{5} \times 22$, the pupil first obtains the product of 22 times 3 fifths as 66 fifths, which, however, he writes in the manner shown herewith, first multiplying 22 by 3 and dividing the product by 5.

$$\begin{array}{r}
879\frac{3}{5} \\
\underline{\times 22} \\
\underline{66} \\
13\frac{1}{5} \\
1758 \\
\underline{1758} \\
29351\frac{1}{5}\ \textit{Ans.}
\end{array}$$

Whether or not there should be drills in the sight multiplication of such combinations as

$$12\tfrac{3}{4} \times 4, \qquad 12\tfrac{2}{3} \times 6, \qquad 12\tfrac{3}{4} \times 8, \text{ etc.,}$$

must depend on the time that can be taken from topics of possibly greater importance at this period of the child's school life. The product of a mixed number and an integer finds frequent place in the after lives of many people, prices often containing fractions of cents—fractions of pounds, yards, etc., being sold; but as this topic will be regularly taught later, the teacher should do no more during the fourth year than is necessary. The space devoted to methods in fractions does not indicate its relative value in the curriculum nor the amount of time that should be given to this topic. The successive steps in the different processes are carefully elaborated, in order to enable a pupil to acquire such an understanding of this subject in its simpler forms as will enable him to add business fractions and to find business products.

Division of fractions, as such, has no place in the primary grades. Although pupils find a number when its half, third, or fourth is given, they do not know they are dividing the given number by $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$. They first find the ratio between 1 and $\frac{1}{2}$, 1 and $\frac{1}{3}$, 1 and $\frac{1}{4}$, to be 2, 3, and 4, respectively, and then use the proper ratio as the multiplier in each case.

While young pupils can be taught to give the correct answer to a series of problems involving the finding of a number when $\frac{3}{4}$ of the number is given, they seldom understand the procedure. About all they know is that certain forms of examples, containing $\frac{3}{4}$ and 12, for instance, require the division of 12 by 4, and the multiplication of the quotient by 3; while others call for the division of 12 by 3, and the multiplication of the quotient by 4. The wording of the analysis likewise tends to confuse them: "If 3 fourths of a number is 12, 1 fourth of the number is 1 third of 12," etc. The brighter ones wonder how 1 fourth can be 1 third, but they continue to repeat the required formula and to obtain the correct result when they happen to guess correctly the type to which the problem belongs. When the problem gives 15 as $\frac{3}{4}$ of the required number, they have no doubts, since it is divisible by 3 and not by 4. If 16 be given as $\frac{3}{4}$ of the required number, the latter will almost invariably be announced as 12, as 16 is not a multiple of 3.

Questions of this kind belong to a higher grade. They should first be employed in written problems solved by the pupils at their seats from the book, and not as a class exercise. Drills in such problems are inadvisable at too early a stage, as they tend to confirm the bad habit of not attending to the conditions of a problem before attempting its solution. The same type of drill is better obtained by the form $15 \div \frac{3}{4}$ rather than by "15 is $\frac{3}{4}$ of what?"

When the course of study requires instruction in division of fractions, the introduction may be made by some such problem as this:

When $1\frac{1}{2}$ yards of cloth cost 15 cents, what is the price of a yard?

If the pupil does not at once indicate the division, he may be led to do so by the question: "How do you find the cost of 1 yard when 3 yards cost

$$\begin{array}{l}1\frac{1}{2})15\text{¢}\\ \hline\end{array}$$

30 cents?" Upon the reply "I divide 30¢ by 3," he is told to show the work, which he readily does. Although he frequently is able to ascertain the answer to the other by indirect means, mentally trying 8¢, perhaps, at first, he should be led to see that the number of yards in the second example and the cost are twice that of the first, and that he can get rid of a fraction in a divisor by multiplying it by the denominator of the fraction, and also that the dividend should be multiplied by the same number. He tests this result in the original problem, 10¢ × $1\frac{1}{2}$, which shows the answer to be correct. Other examples in which the divisor is $1\frac{1}{2}$ may be given, followed by some with $2\frac{1}{2}$, $1\frac{1}{4}$, etc., as divisors.

$$\begin{array}{r} 3\overline{)30\text{¢}} \\ 10\text{¢} \end{array}$$

$$\begin{array}{rr} 1\frac{1}{2}) & 15\text{¢} \\ \times 2 & \times 2 \\ \hline 3\ ) & \underline{30\text{¢}} \end{array}$$

TEST: 10¢ × $1\frac{1}{2}$

A similar method is followed in dividing 15 cents by $\frac{3}{4}$ for instance. The fact should be developed that when the cost of 3 yards, $1\frac{1}{2}$ yards, $2\frac{1}{2}$ yards, $\frac{3}{4}$ yard, etc., is given, the price for 1 yard is obtained by dividing the former by 3, by $1\frac{1}{2}$, by $2\frac{1}{2}$, by $\frac{3}{4}$, etc. He is then made to see that he gets rid of the fractional divisor, $\frac{3}{4}$, by multiplying by 4, following the plan pursued when the divisor is a mixed number.

$$\begin{array}{rr} \frac{3}{4}) & 15\text{¢} \\ 4 & \times 4 \\ \hline 3) & \underline{60\text{¢}} \\ \textit{Ans.} & 20\text{¢} \end{array}$$

A few abstract sight examples may be used to bring to the attention of the pupils the fact that with the same dividend a quotient increases as the divisor decreases. The following may serve:

(*a*) $4\overline{)80}$ (*b*) $2\overline{)80}$ (*c*) $\frac{1}{2}\overline{)80}$ (*d*) $\frac{1}{4}\overline{)80}$

A few others may have fractional divisors, such as $\frac{2}{3}$, $\frac{3}{4}$, etc., to show that with divisors smaller than 1 the quotients are larger than the dividends:

(*a*) $\frac{3}{4}\overline{)12}$ (*b*) $\frac{2}{3}\overline{)12}$ (*c*) $\frac{3}{4}\overline{)24}$ (*d*) $\frac{2}{3}\overline{)24}$

An occasional oral problem may be given that asks what fraction of a pound of 24-cent coffee can be bought for 6 cents, for 12 cents, for 3 cents; what fraction of a yard of 60-cent silk can be bought for 10 cents, for 15 cents, for 30 cents, for 20 cents. The fraction of a yard of 8-cent muslin that can be bought for 1 cent, for 2 cents, for 3 cents, etc., to 8 cents; and the fraction of a foot of 12-cent wire that can be bought for 1 cent, 2 cents, etc., to 12 cents, are useful drills. With large numbers, the problems should require only such answers as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, etc.; with small numbers, the answers may be $\frac{3}{8}$, $\frac{2}{3}$, $\frac{3}{4}$, etc.

Problems of this kind should not be explained or analyzed. They are used to give variety, and the answers are obtained by the inspection of each rather than by any general method. This comes at a later stage.

CHAPTER VII

DENOMINATE NUMBERS

While formal instruction in compound numbers should have no place in the work of the first four years, a child leaving school at the end of this period should be able to solve simple problems involving the common denominate units. This result is obtained through the incidental use of such numbers in the regular work. When, for instance, a child is solving problems containing 2 as a multiplier, one of these may require the finding of the cost of a quart of milk at the rate of 4 cents a pint. This problem differs from the others in furnishing only the multiplicand, it being assumed that the pupils' knowledge of the relation of the pint to the quart will supply the multiplier. When this knowledge has not been derived through experience in buying pints and quarts in errands to the store, it should be obtained in the class-room by actually using a pint measure in filling a quart. When pupils have used a quart measure to fill a gallon, and have learned therefrom the number of quarts the gallon contains, their problems may include under division by 4 the number of gallons of milk that a 40-quart can will hold.

The teacher should confine the early problems to conditions within the experience of the children. In some neighborhoods children frequently sent to the store obtain quite a familiarity with the relationship of certain measures. The teacher takes advantage of this knowledge by giving a number of short oral problems, going the round of the class in a few minutes, having prepared the problems in advance. A first question should not be: "When milk is 8 cents a quart what is the price of a pint?" The children who are accustomed to pay 3 cents a pint give this as their answer, while others, who buy a better variety, state that the price is 5 cents. Even those who give the correct answer may do so because their stores charge 4 cents for a pint, and they announce this number without having regarded the given condition. This may be avoided if the teacher inquires: "Who knows the price of a quart of milk?" A boy who says "Six cents" may be asked the price of a pint. If no other cost is stated, a girl may be asked what would be the price of a pint of milk in winter if a quart then costs 8 cents. Simple problems involving the reduction of pints to quarts, and vice-versa, should be given, being varied to keep up the interest, but such problems should contain only such conditions as are familiar to the pupils, or which may readily be understood. Some children are acquainted with the peck and the bushel; others, with the yard and the foot, or with the foot and the inch. Nothing should be said about denominate numbers, compound numbers, reduction, and the like; nor should any rules be given. Concrete problems throughout the whole four years should employ the words: feet, gallons, yards, pints, bushels, inches, etc., even before children have ascertained the ratio between two related units. The finding of the number of yards in two pieces of cloth, one containing 8 yards and the other containing 9 yards, does not require a knowledge of any table, nor does one that asks the number of bushels of wheat raised on a 40-acre field at the rate of 16 bushels to the acre. Problems of this kind, containing units with which the pupils are familiar, are not found to the necessary extent in the text-books, and the teacher must make them and substitute them for those in the book that are more suitable for a different environment. It is of very great importance in early number work that the conditions of a problem should be so readily understood by a pupil that he can give undivided attention to the determination of the operations required in its solution.

The instruction in weights and measures is begun with the child's earliest lessons in counting. He first employs undetermined units, such as the number of paces in the length of a room and in its width, the number of times a tumbler can be filled from a pitcher of water and the number of pitcherfuls a pail will hold, etc., the character of the work and the time employed varying according to circumstances. Toy money is used, beginning with a number of cents; a nickel is next introduced, with which the pupil combines cents to make a 7-cent purchase, or forms a pile containing a nickel and the number of cents requisite to make it express 8 cents. Later, in succession, may come the dime, quarter-dollar, half-dollar, and the dollar, each being introduced when its use is called for by the stage reached in the number work.

A foot rule may be used to measure short distances, such as the dimensions of the desk, a window, the door, etc., no attention, at first, being paid to its divisions, the approximate distance being stated as so many feet and a little over. Longer distances are measured, approximately, by the yard-stick.

When the time comes for something more exact, the foot divisions of the yard-stick may be used to state that a given distance is 4 yards and a little over 2 feet, etc. The same is done with the foot rule, the length of a desk being announced as 2 feet 6 inches and a little more. The use of the yard-stick and the foot rule gives the number of feet in a yard and the number of inches in a foot. The dimensions of a book, or of an envelope, will naturally introduce the fractions of an inch at the proper time, the half-inch at first, the quarters and the eighths later on. By means of a borrowed tape-line the dimensions of the school building are ascertained, also the size of the yard. The dimensions of the block on which a school is located may interest a city pupil, who finds that the street on which it fronts is 200 feet long between the corners, while the side street measures 600 feet, or thereabouts. In portions of the country containing rectangular farms, the dimensions of a 40-acre lot may be ascertained in rods by means of a 4-rod chain. In the absence of a tape-line, a given distance may be "paced" by different pupils, the length of each one's pace being measured, etc.

It is not supposed that any teacher will have time for all of the work of this kind he might find it advantageous to do, and some teachers may have no time for any of it. It is possible, too, that the pupils of a teacher who neglects this kind of work may be much better taught than the pupils of one that weighs and measures without a definite purpose, and continues this unnecessarily long.

When the school contains a set of liquid measures, the relation between the pint and the quart and that between the quart and the gallon are generally determined by using water or sand, and possibly before these relations are employed in problems. The number of pecks in a bushel and of quarts in a peck may be ascertained by the pupils in the same way. The determination of the number of ounces in a pound by the employment of a pair of scales is unnecessary. The latter may be used to weigh articles of various kinds, but children should be told that there are 16 ounces in a pound instead of requiring them to obtain 16 one-ounce weights to ascertain this for themselves. When time is readily obtainable, pupils of the proper age might receive a certain amount of benefit by estimating the weight of different objects held in the hand for a moment, and testing their estimates by the use of the scales; but work of this kind should be done after school by pupils interested enough to remain for that purpose.

The fact that a pint of water weighs about a pound may be ascertained; the weight of a bushel of oats may be calculated from the weight of a pint (dry measure) and the difference found between the calculated weight and that of the legal bushel; the difference between the weight of a quart of sand (liquid measure) and that of a quart (dry measure), etc.

When the course of study prescribes the denominate units to be taught and the grade in which each is introduced, the character of the problems must be determined by the attainments of the pupils.

The pint, the quart, and the gallon frequently constitute the first set. The earliest problems should deal exclusively with the pint and the quart, which lend themselves to the use of 2 as a multiplier and a divisor. The relation between the quart and the gallon should not be employed in problems until 4 is reached. Even when pupils have used 4 as a multiplier and a divisor before the introduction of the quart and the pint, problems involving the quart and the gallon should be postponed for a time in order that the pupils may be accustomed to the first ratio before dealing with a new one. Reductions in this grade should require but one step, quarts to pints or gallons to quarts, not gallons to pints. An oral problem asking the number of pints in 1 gallon requires but a single step, one asking the number of pints in 2 gallons requires two steps for its solution.

While a teacher should not hesitate to go beyond the requirements of his grade, he should have a reason for doing so. The time allotted to arithmetic, although often excessive so far as the importance of the subject is concerned, is frequently too short for the details provided in many overloaded courses of study, and the thoughtful teacher should aim at intensive rather than at extensive treatment of the subject.

Problems involving two steps, which belong to the latter half of the third year, may consist of the reduction of gallons to pints, of quarts and pints to pints, etc. The preliminary preparation of the pupils required in teaching a new process is not called for in any of the denominate number work of the primary grades. All that is necessary is to keep the numbers small enough and the conditions few enough. A pupil at this stage can find the number of quarts in 4 gallons 3 quarts, and the number of gallons and quarts in 19 quarts. The form in which the answer to the latter is written may need to be shown, viz., 4 gal. 3 qt. In previous exercises the pupils have learned the customary abbreviations of such denominate numbers as their problems contained; at this point the compound number is written for the first time. If a pupil pluralizes the abbreviation, he should be told that a final s is superfluous.

Other two-step problems call for the cost of 3 quarts of oil when it is sold for 16 cents a gallon, or that of a quart and a pint of ice-cream at 30 cents a pint. As the price of a gallon is not always four times the cost of a quart, the teacher may find it useful to give a problem asking how much less is paid for a gallon at 25 cents than in buying it at the rate of 7 cents a quart.

Nearly all the problems in denominate numbers should be oral. In the business world the written ones are very few, not nearly so many as to require the space devoted to this subject in the average text-book. The first occasion for using the pencil might come with the reduction of 5 pounds to ounces, in the appropriate grade, and followed immediately by the reduction of 5 pounds 9 ounces to ounces, the teacher making no allusion to the connection between them. When this problem is worked on the board by a pupil, the teacher should accept any arrangement presented, without suggesting the form regularly employed in reduction exercises, which may well be deferred for a year or two.

Examples in compound numbers should always, in primary grades, be restricted to two successive units; within which requirement the reduction of gallons and pints to pints does not come.

If the work begins in the third year, it generally includes the following four tables, two for each of the two last years of the primary course, the order in which they are taken up depending upon local conditions:

Pint, quart, gallon; ounce, pound; inch, foot, yard; quart, peck, bushel.

At the proper time, the denominate units already known lend themselves to an occasional oral exercise in some forms of fraction work, which should, however, be carefully limited to conform to other requirements. The following are types:

How many inches in (*a*) $\frac{1}{2}$ ft.? (*b*) $\frac{1}{4}$ ft.? (*c*) $\frac{1}{3}$ ft.? (*d*) $\frac{2}{3}$ ft.? (*e*) $\frac{3}{4}$ ft.?

What part of a foot is (*a*) 6 in.? (*b*) 3 in.? (*c*) 4 in.? What part is (*a*) 9 in.? (*b*) 8 in.?

How many ounces in (*a*) $\frac{1}{2}$ lb.? (*b*) $\frac{1}{4}$ lb.? (*c*) $\frac{1}{8}$ lb.? In (*a*) $\frac{3}{4}$ lb.? (*b*) $\frac{3}{8}$ lb.? (*c*) $\frac{5}{8}$ lb.? (*d*) $\frac{7}{8}$ lb.?

What part of a pound is (*a*) 8 oz.? (*b*) 4 oz.? (*c*) 2 oz.? What part is (*a*) 6 oz.? (*b*) 10 oz.? (*c*) 12 oz.? (*d*) 14 oz.?

No rule should be given for the change of 9 inches to the fraction of a foot, 10 ounces to the fraction of a pound, etc. The pupil should be permitted to do these in his own way. A problem involving the cost of 4 ounces or of 8 ounces of an article whose price is 12 cents, 24 cents, etc., per pound may be given for the sake of variety, to show pupils that it is unnecessary to first obtain the cost of 1 ounce, etc.

CHAPTER VIII

AREAS AND VOLUMES

THE applications of addition, subtraction, etc., constitute what are known in the text-books as problems. In the lower grades these should exclude all types containing conditions that are not fully understood without special explanation. Gradually, the pupils become able to determine, from increasing experience and through the terms of the problem, the operations required for its solution.

The aim of the problem being to develop a child's reasoning ability, its terms do not state directly what operation is called for, and no rule should be given for the solution of any type.

A class of problems called measurements was formerly postponed in the text-books until the end of the elementary course. On account of its importance, however, it was brought forward, in its simpler applications, to the latter half of the sixth year. Later, it became one of the features of the earliest grammar grade work. At present, owing possibly to the growth of instruction in form, primary pupils are expected to ascertain the areas and perimeters of rectangles.

The method of treatment will depend somewhat on the lessons in form. If the pupils have handled one-inch cubes, and have cut paper into one-inch squares, they have a certain familiarity with the square inch, and they are ready to apply this as a unit of measure to a rectangle of convenient size, say 3 inches by 2 inches. By drawing the rectangle and covering the surface with one-inch paper squares they are ready to state that the rectangle contains two rows, each consisting of three 1-inch squares. A rectangle 5 inches by 3 inches may be drawn on the blackboard, 1-inch divisions marked on each side, and the rectangle divided into 1-inch squares. A few such examples develop the fact that the number of square inches in a rectangle is equal to the product of the number of inches in its length by the number in its width. In some schools these incidental lessons extend over several terms, rectangles subdivided into equal squares being sometimes used in the multiplication exercises to aid in developing an acquaintance with some products.

In a class taking up areas of rectangles without preliminary instruction in form, a teacher might draw on the blackboard a rectangle 3 convenient units in length and 2 in width, requiring the children to draw one on their papers at the same time, 3 inches long and 2 inches wide. The teacher lays off the unit divisions on the length and on the opposite side, and calls upon the children to do the same. He then lays off the width and its opposite side in the same way, which the children likewise do. The rectangles are then subdivided into squares, those of the pupils' drawings being 1-inch squares. The children are next told that each square is called a square inch, and different pupils are called upon to tell how many square inches their figures contain. The members of the class may then be asked to draw a rectangle, say 4 inches by 3 inches, and to divide it into 1-inch squares. The word rectangle may be used without any explanation of its meaning, as the children take it for granted that they are to draw a figure resembling the previous one. Passing among

the pupils during the drawing, the teacher calls attention to the need of a square corner, when a pupil's drawing is seriously astray.

In this, as in all other forms of objective illustration of number work, the drawing of rectangles should be continued only just so long as is necessary to enable children to get an idea of a square inch, and the fact that the number of square inches in any rectangle is equal to the product of the number of inches in one dimension by the number in the other. It is not advisable that they should state this; their knowledge of the fact being shown by the answer to questions such as the following, the numbers in each being different:

(*a*) "If you should draw a rectangle 4 inches by 5 inches, how many square inches would it contain?"

(*b*) "What is the area of a rectangle 6 (7, 8, 9) inches long and 4 (5, 6, 7) inches wide?"

The pupils should learn the word "area" through its employment by the teacher, rather than by a definition, and the word "dimensions" should become a part of their vocabulary in the same way. When the area of a rectangle 5 inches by 5 inches is called for, the class may be asked if any pupil knows of another name for a rectangle when its sides are equal. After pupils find the length of a fence enclosing a plot 20 yards long and 10 yards wide, they may be told that "perimeter" is one word to express the same thing as "distance about," the former being thereafter used in all the teacher's questions.

The pupils may measure the dimensions of one side of their books to determine the number of square inches of cloth required to cover each side. To avoid fractions at the start, they are told to take the next whole number of inches, as an allowance for overlapping. If some of the illustrations in their readers have dimensions in even inches, the finding of the cost of the plate at 20 cents, for instance, a square inch, might be employed in a problem.

The square foot should be illustrated by drawing on the blackboard a rectangle of convenient dimensions. Attention should not be called to the connection between the square inch and the square foot, such as the number of square inches in the latter. A square should also be drawn, having sides a yard in length, to give an idea of a square yard.

Primary problems should express the dimensions in linear units corresponding to the square units required in the answers. An example should not be given at this stage asking the number of square yards, for instance, in a carpet 24 feet long, 18 feet wide.

When, however, such problems are required, the children should be taught first to change the given linear units into those corresponding with the square units called for in the result. The former should be such as to require no work more difficult than the multiplication of a mixed number by an integer.

If necessary, the product of an integer and a mixed number may be illustrated by such a rectangle as 4 inches by $2\frac{1}{2}$ inches, drawn by the pupils, which shows them that half square inches are combined into whole ones in stating an answer. The area of a $2\frac{1}{2}$-inch square may be obtained objectively as $6\frac{1}{4}$ square inches; but, as a rule, examples of this kind belong to the period at which multiplication of fractions finds a place in the course of study. A rectangle 2 inches by $\frac{1}{2}$ inch is useful to enable pupils to understand that a surface may contain a square inch without being a one-inch square.

The method of finding the volume of a rectangular solid may be illustrated by forming one with 24 one-inch cubes, the base consisting of 4 rows of 3 cubes each, and 2 equal tiers of these forming the solid. The same 24 blocks may be used to form other rectangular prisms; as, for instance, one having a base 2 inches by 4 inches, and 3 inches high; one 2 inches by 3 inches, and 4 inches high; etc. The pupils readily understand that the volume in cubic inches is equal to the combined product of the three dimensions expressed in inches. Only the simplest applications should be made. Some classes, familiar with wood in cords, might find the volume of a pile of wood measuring 8 ft. by 4 ft. by 4 ft. Possibly the weight of a rectangular piece of ice 3 ft. by 2 ft. by 1 ft. at 60 pounds to the cubic foot would not be too difficult for pupils accustomed to see artificial ice in cakes of these dimensions. The teacher must bear in mind, however, that regular work of this kind belongs to a grade two years in advance, and that he should do only so much of it as is called for by the course of study.

CHAPTER IX

MISCELLANEOUS CALCULATIONS

When bills are taught during the fourth year, the teacher should be guided by local conditions in determining the form to employ. Bills are of two general types, the one used by the store-keeper and the one used by the mechanic. The former are generally made out on regular bill-heads, the first line containing the name of the locality, with a space left for the date. The second line is left blank, on which to write the name of the purchaser. The third line is printed, and gives the name of the selling individual or firm. Below comes the space for the enumeration of the items sold, with the quantities, prices, extensions, and footing. If the school contains a mimeograph, the teacher selecting this type should prepare a number of bill-heads with the customary ruling, in the following form:

FAYETTEVILLE, ILL.,   191—

M________________

Bought of STEPHEN T. HOILE & Co.
General Merchandise

As the other work of the fourth grade leaves no spare time for ruling lines, etc., and as there should be only one style of bill presented to the pupils at this time, the second type is the more valuable, especially as it can be used by a store-keeper also. The following is a bill of this kind made out and receipted:

LAMBERTVILLE, WIS., Sept. 29, 1911.

MR. HUNTER COLLINS

TO WILLIAM F. ZIEGLER, DR.

To .. 3 days' Labor	$2.75	$8.25
.....12 Roses	.25	3.—
.....45 Hyacinths	.06	2.70
		$13.95

Received payment,
Oct. 5, 1911.
William F. Ziegler.

In this form the third line contains the name of the person to whom the money is due, preceded by the word "To" and followed by the abbreviation for debtor, the two lines meaning that Mr. Collins is indebted to Mr. Ziegler; and the items follow. As this bill does not require the ruling of any lines, one can readily be made out in a portion of the time allotted for an arithmetic recitation.

A sample bill is written on the blackboard by a pupil, under the direction of the teacher, who elicits from different pupils the fact that the heading of the bill contains three lines, one for the place and date, one for the name of the debtor, and one for the name of the creditor. When the trade of the latter is selected, pupils suggest the daily wage rate, also the prices of such articles as he is likely to supply. The extensions and the footing may be omitted, but the receipt should be written. The class may then copy it from the blackboard, supplying the omissions.

The teacher should take advantage of the exercises in writing bills to give practice in finding the product of two numbers written on a line. For this purpose the pupils should be limited to a single sheet of paper, not being permitted the use of a second piece on which to make side calculations. These should be rendered unnecessary by the careful selection of the quantities and prices given by the teacher. These should cover the various factors employed in the so-called "blackboard exercises" in which the pupils write only the answers, one figure at a time. Combinations similar to the following may be employed:

(*a*) 12 @ $\$\,.18\frac{3}{4}$
(*b*) $146\frac{1}{2}$ @ $\$\,.04$
(*c*) 24 @ $\$1.22$
(*d*) 31 @ $\$.33$
(*e*) 496 @ $\$.25$
(*f*) 375 @ $\$.50$

In (*a*) and (*c*) 12 and 24, respectively, are used as multipliers; while in (*b*) $146\frac{1}{2}$ is multiplied by 4. In (*a*) and (*b*) the product of 12 by $\frac{3}{4}$ and that of $\frac{1}{2}$ by 4 are added in with 12 times 8 and 4 times 6, respectively. The combinations (*e*) and (*f*) are usable only when the pupils have learned to employ $\$\frac{1}{4}$ and $\$\frac{1}{2}$ for 25¢ and 50¢, respectively, in which case they divide 496 by 4 and 375 by 2, writing the first figure in the place for hundreds of dollars.

Each bill should be limited to perhaps four items, and the numbers should not be unnecessarily large. An occasional combination should employ numbers greater than 12 for both factors, as in (*c*) and (*d*), to show pupils that a product by 24 or by 33 may be written at sight. The teacher will, of course, insist upon the omission of the unnecessary @ in a bill, as well as the limitation of the dollar mark ($\$$) to the first line and the result.

Bills for articles purchased may be made out in the second form when mimeographed blanks are not available. If the latter are used, the extension of each item is placed in the first of the two sets of columns for dollars and cents, the footing being written in the second set. Both sets are generally found in all printed forms, but the second at this time is used merely for the footing.

As a preliminary to the dictation of the items, the teacher directs the class to prepare for a bill, which the pupils do by writing the place and the day of the lesson on the first line, each writing his own name for the second line, and, possibly, the teacher's name for the third line. All this is done promptly. Then the items are dictated, after which the pupils are allowed a few minutes to write the extensions, the footing, and the receipt, the teacher walking around the class to see that no side work is done, etc.

After a reasonable time the pupils are directed to lay aside their pens, and the results are obtained in the manner previously outlined. The footing may first be called for, and the correct number written on the blackboard by the scorer. The several extensions are then separately taken up, and the totals noted. A pupil having difficulty with one of the combinations may work it aloud at the blackboard; or all of the combinations may be written on the board and different pupils, who have made mistakes, announce the successive numbers in the products, the latter being written by the scorer as each is given.

Exercises substituting fractions of a dollar for cents in multiplication of United States money, if used at all, should be limited, in this grade, to the aliquot parts, and to the simplest of these; $\$\frac{1}{2}$ for 50¢, $\$\frac{1}{4}$ for 25¢, and $\$\frac{1}{3}$ for $33\frac{1}{3}$¢ being sufficient at this time. The first oral exercises should consist of such questions as these, which are preceded by a few oral problems as a preliminary:

How many dollars in (*a*) 10 half-dollars? (*b*) 20 half-dollars? (*c*) 16 half-dollars? (*d*) 18 half-dollars? etc.

This may be followed by such examples as:

How many dollars and cents in (*a*) 9 half-dollars? (*b*) 21 half-dollars? (*c*) 25 half-dollars? etc.

The corresponding sight drills will consist of such combinations as:

(*a*) \$.50 × 24	(*b*) \$.50 × 42	(*c*) \$.50 × 84
(*d*) \$.50 × 25	(*e*) \$.50 × 17	(*f*) \$.50 × 23
etc.	etc.	etc.

The blackboard exercises may contain multipliers of several figures, the pupils writing one figure of the answer at a time, and obtaining the latter by dividing the multiplier by 2. The remainder, if any, should be written as cents.

A similar procedure is followed using \$¼ or 25¢ as the multiplicand, the first multipliers being exactly divisible by 4. When, later, these are not multiples of 4, the pupils should learn to connect 25¢ at once with a remainder of 1, 50¢ with a remainder of 2, and 75¢ with a remainder of 3.

If 33⅓¢ is one of the aliquot parts to be taught, it need not be dwelt upon to the same extent as the others, being less frequently used in subsequent transactions. After each has had its own drills, miscellaneous exercises embracing all three should be used in oral and sight exercises and problems. Short methods of multiplying by 25 and 33⅓ may well be postponed for a later grade.

Two-step problems involving multiplication and division constitute an important type throughout the course. In the primary grades, they should be given with other two-step problems without calling special attention to them, the pupil being permitted to do each problem in his own way. The first ones of this kind should require, say, the finding of the cost of 5 articles when the cost of 2 is given, the pupil finding from the latter the cost of 1 by dividing the given cost by 2, and multiplying the quotient by 5, to obtain the result. In this form, the given cost is generally exactly divisible by the number of articles bought.

During the fourth year, the oral problems should be somewhat different; as is shown, for instance, in the following:

(*a*) When 3 acres of land yield 44 bushels of wheat, how many bushels should 6 acres yield?

(*b*) What is the cost of a dozen oranges at the rate of 3 for 5 cents?

(*c*) If 5 men require 18 hours to do a piece of work, how long should it take 10 men to do the same work?

The pupils that in (*a*) begin by endeavoring to find the yield of 1 acre do not reach the result in the allotted time. If one of these is called upon to give his analysis, and proves unable to find the quotient of 44 by 3, etc., he is told that the problem lends itself to a simpler method of solution. This is furnished by one of the pupils who employed it.

$$\frac{295 \text{ bu.} \times \overset{2}{\cancel{28}}}{\cancel{14}}$$

Very few written problems should be given to be solved by the latter method. Later on, when cancellation is practised, the preliminary indication of the required operations takes the same form in each. Thus, the number of bushels raised on 28 acres when 14 acres yield 295 bushels is indicated: 295 bu. × 2, in which the ratio of the acres is obtained by canceling 14 and 28. When 180 bushels is given as the yield of 15 acres, from which to calculate the yield of 17 acres, the cancellation takes in 15 with 180 bushels, thus:

$$\frac{\overset{12}{\cancel{180}} \text{ bu.} \times 17}{\cancel{15}}$$

At this stage the pupils possibly indicate division chiefly by the sign (÷), instead of writing it as a denominator; or by the hook, as is shown in the following plan, which might be adopted by some pupil for the solution of the second example:

$$\begin{array}{rl}
 & \overline{12\text{ bu.}} \times 17 = 204\text{ bu.}\quad \textit{Ans.}\\
15) & 180\text{ bu.}\\
 & \underline{15}\\
 & 30\\
 & \underline{30}
\end{array}$$

The following is a method that might possibly be employed by a pupil in the solution of the first. In this he indicates his reason for the use of 2 as a multiplier.

$$\begin{array}{rr}
 & 295\text{ bu.}\\
(28\text{ A.} \div 14\text{ A.}) & \underline{\times 2}\\
\textit{Ans.} & 590\text{ bu.}
\end{array}$$

While model methods of arranging the work in solving problems should not be prescribed, the teacher encourages other members of the class to point out mistakes made in a solution presented on the board by a pupil. The former should not at this stage expect the children to indicate the operations in the manner they will follow later on, when they will have learned the sequence of signs, the use of marks of aggregation, etc.

In order that the teacher may see all of the work of a pupil, no paper, other than the sheet handed in, should be used for side calculations. Erasures should not be made, a line being drawn through a mistake and the correction written alongside. No time should be spent in rewriting a solution on a clean sheet to be presented, thus preventing the teacher from ascertaining the entire procedure of a pupil, and the types of blunders he makes.

THIRD PART — FIFTH AND SIXTH YEARS

CHAPTER I

INCIDENTAL REVIEWS

Although at the end of the fourth year the child has had some instruction in adding and subtracting easy fractions, in multiplying by a mixed number, and in working simple problems involving the more familiar denominate units, his time has been devoted chiefly to the development of his ability to perform the fundamental operations with simple numbers.

During the next two years the attention of the pupils is given chiefly to fractions, common and decimal, and to compound numbers. These now receive systematic treatment, reduction, addition, subtraction, multiplication, and division of each being taken up in turn as the major topic.

Exclusive attention is not, however, given to the latter. When formal instruction in common fractions is the chief work, incidental treatment of decimals, compound numbers, and measurements constitutes a feature of the course.

When decimals become the major topic, then percentage applications are introduced. These are extended somewhat during the period especially devoted to compound numbers, and the mensuration of rectangular surfaces and volumes receives adequate treatment.

The end of the sixth year marks the completion of the second stage of the elementary course, inasmuch as it gives the pupils a reasonable experience in process work with all types of arithmetical numbers and a working acquaintance with the most important applications.

Throughout the fifth and sixth years the pupil should incidentally review his earlier work in the fundamental operations from the standpoint of his greater development. When three-place decimals are taught in the fifth year, an opportunity is offered for a clearer insight into Arabic notation and numeration, and its extension downwards to tenths, hundredths, thousandths, etc.

The separation of a large number into its orders, periods, etc., should receive due consideration. Practice in the numeration and notation of integers and of decimals will be generally obtained by means of the written exercises.

Besides the weekly review, which should include at least one long example in addition, there should be some blackboard work with short exercises in which the pupils are required to show the reason for the arrangement of addends, the explanation of carrying, etc. Examples in addition should include a few tabular forms, involving the finding of the sums of lines and of columns, with the verification of the results by means of the grand total. The following table, giving the attendance by grades on each day of the week, is a type. From this, the weekly aggregate attendance of each grade and of the entire school, as well as the total attendance for each day, is to be found.

GRADES	MON.	TUES.	WED.	THURS.	FRI.	WEEKLY AGGREGATE
1st grade........	42	41	43	43	42	
2d "	39	39	40	39	39	
3d "	38	38	37	36	37	
4th "	36	35	34	34	36	
5th "	32	30	31	30	32	
6th "	28	28	28	27	28	
7th "	24	22	24	24	24	
8th "	20	20	20	20	20	
Daily totals						

The oral exercises and problems in addition should gradually increase in difficulty throughout the middle grades, the following types showing the development:

5th year 1st term	57 + 86	84 + 98	39 + 77
5th year 2d term	169 + 25	49 + 129	119 + 78
6th year 1st term	167 + 85	69 + 138	145 + 78
6th year 2d term	267 + 85	69 + 338	445 + 78

In combining these numbers the pupil omits the word "hundred," thinking (or saying), when adding 57 and 86, one thirty-seven, one forty-three. The words dollars and cents are also omitted in combining \$2.67 + \$.85, these amounts being considered merely as two sixty-seven and eighty-five, of which the successive sums are three forty-seven and three fifty-two. In giving the answer, however, the result is stated as three dollars fifty-two cents.

When larger numbers are used, the combinations should be less difficult. The following types are suggested for the four terms:

I.	*(a)* \$12.25 + 1.54	*(b)* \$5.40 + 3.60	*(c)* 6400 + 2400	*(d)* 28,000 + 9,000	*(e)* 22,000 42,000
II.	*(a)* \$21.40 + 15.30	*(b)* \$4.70 + 5.80	*(c)* 5400 + 6500	*(d)* 28,000 + 19,000	*(e)* 17,000 + 6,500
III.	*(a)* \$21.40 + 45.30	*(b)* \$14.70 + 5.80	*(c)* 5480 + 6500	*(d)* 32,000 + 29,000	*(e)* 18,600 + 7,200
IV.	*(a)* \$21.48 + 45.30	*(b)* \$14.75 + 5.80	*(c)* 5480 + 6520	*(d)* 32,800 + 29,000	*(e)* 18,600 + 7,500

The combinations in *(c)* should be read as hundreds, 64 hundred + 24 hundred, etc. The ability to handle large numbers frequently prevents mistakes in written work by enabling pupils to compare the result with an approximate one obtained by using round numbers.

Some written examples may consist of longer exercises in adding the fractions and denominate numbers previously studied in the lower grades.

In adding long columns the pupils should write the total of each column at one side, in order that it may not be necessary to go back to the first column when the second addition of one of the later ones gives a total different from the one first obtained. When the side totals are available, the pupil can begin with the next column to the right, using the carrying figure given in the sum of the preceding column. If a sum is first found by adding upwards, the result is checked by adding downwards.

$ 18.75	
315.00	
78.89	
26.74	50
.88	59
5.43	56
247.76	29
98.89	7
4.56	
$796.90	

Subtraction examples should be worked by successive pupils at the blackboard with a full explanation of each step. When the Austrian method is used, this analysis needs only to explain that the remainder is a number by which the subtrahend is increased to produce the minuend. The correctness of the result is determined by combining the remainder and the subtrahend, adding upwards in testing a difference found by the Austrian method.

An oral problem in subtraction may employ the sum in each different type suggested for addition as the minuend, and one of the given numbers as the subtrahend.

$$\begin{array}{r} \$100.00 \\ \left.\begin{array}{r} 8.63 \\ 52.84 \\ 6.19 \end{array}\right\} \\ \hline \$\ 32.34 \end{array}$$

Some examples in written subtraction may involve the deduction of the sum of several numbers from a minuend in a single operation, if subtraction is taught by the Austrian method. Thus, the remainder left when $100 is diminished by the sum of $8.63, $52.84, and $6.19, is written at once, the pupil saying:

13 (9 + 4), 16 (adding 3), and *4* are 20. Write 4.

3 (carrying 2), 11 (adding 8), 17 (adding 6), and *3* are 20. Write 3.

8 (carrying 2), 18 (adding 2 and 8), and *2* are 20. Write 2.

7 (carrying 2) and *3* are 10. Write 3.

A pupil working a multiplication example at the blackboard should explain the meaning of each partial product, and give the reason for placing the right-hand figure of each under the figure of the multiplier that produces it. The reason for the placing of the cipher in the ones' place in the accompanying example should be required. Before beginning the work a pupil should be asked the number of figures in the product, which is equal to the combined number in the multiplier and multiplicand, or one less. After (*a*) is worked, the product of 4783 by 24 may be found, and the steps compared.

$$\begin{array}{rr}
(a) & 4\,783 \\
 & \underline{\times\ 2\,040} \\
 & 191\,320 \\
 & \underline{9\,566} \\
\textit{Ans.} & 9{,}757{,}320
\end{array}
\qquad
\begin{array}{rl}
(b)\ 4783 & \\
\underline{\times\ 24} & \\
19132 & \\
\underline{9566} & \\
114792 & \textit{Ans.}
\end{array}$$

Some practice should be had in multiplying by such numbers as (*a*) 19 and (*b*) 91, without rewriting the multiplicand. This is avoided by placing the multiplier at the right of the multiplicand instead of underneath, the latter being then used as one of the partial products. Products by (*a*) 99 and (*b*) 999 should be obtained by subtracting the multiplicand from its product by 100 and 1000, respectively. The latter can be placed above or below the original number. After the pupils have had some practice with aliquot parts of a dollar, products by 25, $12\frac{1}{2}$, $33\frac{1}{3}$, $16\frac{2}{3}$, etc.,

$$\begin{array}{rl}
 & 4284 \\
(a) & \underline{476 \times 19} \\
 & 9044\ \textit{Ans.}
\end{array}
\qquad
\begin{array}{rl}
(b) & 476 \times 91 \\
 & \underline{4284} \\
 & 43316\ \textit{Ans.}
\end{array}$$

$$\begin{array}{rl}
(a) & 476 \times 99 \\
 & \underline{47600} \\
 & 47124\ \textit{Ans.}
\end{array}
\qquad
\begin{array}{rl}
 & 476000 \\
(b) & \underline{476} \times 999 \\
 & 475524\ \textit{Ans.}
\end{array}
\qquad
\begin{array}{l}
\underline{1847 \times 25} \\
46175\ \textit{Ans.}
\end{array}$$

may be written under the multiplicand, as shown above. The quotient by 4 is written underneath, and the 3 remainder is changed to 75. An occasional example should be given in which the product is obtained by employing the factors of the multiplier. In the accompanying illustration the product is obtained at (*a*) by taking 7 as the first multiplier and multiplying the resulting product by 8. This result is tested at (*b*) by reversing the order of the factors.

(*a*) $\begin{array}{r} 476 \times 56 \\ \times 7 \\ \hline 3332 \\ \times 8 \\ \hline 26656 \text{ } \textit{Ans.} \end{array}$ (*b*) $\begin{array}{r} 476 \times 56 \\ \times 8 \\ \hline 3808 \\ \times 7 \\ \hline 26656 \text{ } \textit{Ans.} \end{array}$

Oral problems in multiplication should be gradually extended to cover the product of a number of two orders by a number of one order; such as:

I. (*a*) 27 × 7 (*b*) 84 × 9 (*c*) 56 × 8 (*d*) 95 × 6

In multiplying 27 by 7, the pupil should first think 7 times 20 (140), and to this add 7 times 7 (49); 8 times 56 equals 8 times 50 (400) + 8 times 6 (48).

Larger multiplicands should also be employed occasionally, but the combinations should decrease in difficulty as the size of the multiplicand increases. When the latter consists of a number of three orders, the product should involve a minimum of carrying:

II. (*a*) $\begin{array}{r} 310 \\ \times 8 \\ \hline \end{array}$ (*b*) $\begin{array}{r} 420 \\ \times 4 \\ \hline \end{array}$ (*c*) $\begin{array}{r} 205 \\ \times 9 \\ \hline \end{array}$ (*d*) $\begin{array}{r} 807 \\ \times 6 \\ \hline \end{array}$ (*e*) $\begin{array}{r} 530 \\ \times 3 \\ \hline \end{array}$ (*f*) $\begin{array}{r} 609 \\ \times 5 \\ \hline \end{array}$

The first multiplicands of more than three orders should consist of thousands:

III. (*a*) $\begin{array}{r} 7{,}000 \\ \times 9 \\ \hline \end{array}$ (*b*) $\begin{array}{r} 12{,}000 \\ \times 7 \\ \hline \end{array}$ (*c*) $\begin{array}{r} 24{,}000 \\ \times 2 \\ \hline \end{array}$ (*d*) $\begin{array}{r} 32{,}000 \\ \times 3 \\ \hline \end{array}$

It is important that pupils should obtain the ability to think in large numbers provided the combinations are readily made. All of the foregoing types should be contained in oral problems, those in set II being frequently used as dollars and cents.

The arrangement of the factors in multiplication is immaterial. Many teachers write a terminal cipher beyond the figures of the multiplicand as in (*a*), in order to ensure the bringing down of the cipher. The pupils are less likely to be confused by placing the right-hand figures (integral or decimal) of both factors in a row, permitting only a common fraction in either to extend beyond. By this arrangement figures of the same order will be in the same row throughout.

$$(a)\quad \begin{array}{r} 36 \\ \times 30 \\ \hline 1080 \end{array}\ \textit{Ans.} \qquad\qquad (b)\quad \begin{array}{r} 36 \\ \times 30 \\ \hline 1080 \end{array}$$

Sight exercises should be given in finding products by 25, 99, etc., after these multipliers have been used in the written work. The first examples should be simple, such as:

(*a*) 24 × 25 (*b*) 25 × 48 (*c*) 36 × 25 (*d*) 25 × 28

In these the pupils think of the result as $\frac{1}{4}$ of 24 hundred, 48 hundred, 36 hundred, and 28 hundred, respectively.

Blackboard exercises containing larger multiplicands should frequently be employed, the pupils writing only the products. The multiplicands may consist of numbers of any convenient size, as the pupils write but one figure of the result at a time. The following are typical:

(*a*) 21648 × 25 (*b*) 21649 × 25 (*c*) 21651 × 25

In finding mentally products by 99, the simplest combinations are 99 times 2 to 9, and 99 times 90 to 99. Thus 99 times 4 is 4 hundred minus 4, and 99 times 96 is 96 hundred minus 96, the subtraction being equally simple in each. Other multiplicands of two figures should then be employed.

When a pupil working at the blackboard employs a circuitous plan of doing his work, the teacher should inquire if any other pupil can suggest a shorter way. Blackboard exercises the answers to which can be written only when a short method is used, are useful in suggesting to pupils possible ways of shortening their written work. Thus in obtaining the continued product of 8 × 32 × 5, the pupil who multiplies 8 by 5 has only to write the product of 32 by 40. Some examples of this type are:

(*a*) 8 × 37 × 25 (*b*) 33 × 75 × 4

The multiplication reviews may include products in which one factor is a mixed number, in case these have already been taught in a lower grade. Some of these mixed number multipliers should have the numerator of the fraction the same number as the integer. In multiplying 457 by $3\frac{3}{5}$, for instance, the product of 457 by 3 is divided by 5 without drawing a line under it, as at (*a*), and the two partial products are combined, saving one row of figures over the method shown at (*b*).

$$\begin{array}{rrl}(a) & 457 & \\ & \times 3\frac{3}{5} & \\ \hline & 1371 & \text{product by } 3 \\ & 274\frac{1}{5} & \text{``} \quad \text{``} \quad \frac{3}{5} \\ \hline \textit{Ans.} & 1645\frac{1}{5} & \text{product by } 3\frac{3}{5}\end{array}$$

$$\begin{array}{rr}(b) & 457 \\ & \times 3\frac{3}{5} \\ \hline & 5)\underline{1371} \\ & 274\frac{1}{5} \\ & 1371 \\ \hline & 1645\frac{1}{5}\end{array}$$

A few examples should follow the method employed in multiplying by 99. Thus the product by $9\frac{7}{8}$, for instance, is obtained by deducting $\frac{1}{8}$ of the multiplicand from 10 times the multiplicand, as shown at (*c*), which is shorter than the usual method, which is given at (*d*).

$$\begin{array}{rrl}(c) & 184 & \\ & \times 9\frac{7}{8} & \\ \hline & 1840 & (10 \text{ times } 184) \\ (\text{Deduct}) & 23 & (\frac{1}{8} \text{ of } 184) \\ \hline \textit{Ans.} & 1817 & (9\frac{7}{8} \text{ times } 184)\end{array}$$

$$\begin{array}{rrl}(d) & 184 & \\ & \times 9\frac{7}{8} & \\ \hline & 8)\underline{1288} & \\ & 161 & \\ & 1656 & \\ \hline & 1817 & \textit{Ans.}\end{array}$$

The following are typical sight examples, to which this method is applicable:

(*a*) $32 \times 9\frac{3}{4}$ (*b*) $32 \times 19\frac{7}{8}$ (*c*) $32 \times 24\frac{7}{8}$

and also these:

(*d*) $16 \times 99\frac{1}{2}$ (*e*) $16 \times 99\frac{3}{4}$ (*f*) $16 \times 99\frac{7}{8}$

While the foregoing are not particularly practical, they are useful in showing the older pupils that the same result may be obtained in different ways. A short method should not, however, be presented until after the pupils have become proficient in performing operations by the general one.

In order that pupils may acquire the practice of "adding in" the numerator of the fraction in reducing a mixed number to an improper fraction, the reviews in multiplication may include mixed number multiplicands among the blackboard exercises, the pupils writing only the answers.

I. (*a*) $28\frac{3}{4}$ (*b*) $31\frac{7}{8}$ (*c*) $52\frac{3}{5}$ (*d*) $93\frac{2}{3}$ (*e*) $37\frac{5}{6}$
$\underline{\times 4}$ $\underline{\times 8}$ $\underline{\times 5}$ $\underline{\times 3}$ $\underline{\times 6}$

As the next type, similar multiplicands may have as multipliers numbers that are multiples of the denominators:

II. (*a*) $28\frac{3}{4}$ (*b*) $31\frac{7}{8}$ (*c*) $52\frac{3}{5}$ (*d*) $93\frac{2}{3}$ (*e*) $37\frac{5}{6}$
$\underline{\times 8}$ $\underline{\times 16}$ $\underline{\times 15}$ $\underline{\times 9}$ $\underline{\times 12}$

The remaining type includes any simple multiplier:

III. (*a*) $28\frac{3}{4}$ (*b*) $31\frac{7}{8}$ (*c*) $52\frac{3}{5}$ (*d*) $93\frac{2}{3}$ (*e*) $37\frac{5}{6}$
$\underline{\times 6}$ $\underline{\times 5}$ $\underline{\times 7}$ $\underline{\times 8}$ $\underline{\times 4}$

The aliquot parts of a dollar should be employed in many exercises and problems. The following are some types:

I. Reduce to cents: (*a*) $\$\frac{1}{2}$, (*b*) $\$\frac{1}{4}$, (*c*) $\$\frac{1}{3}$, etc.

II. Find the cost of 24 articles at:

(*a*) $\$\frac{1}{4}$, (*b*) $\$\frac{3}{4}$, (*c*) $\$\frac{2}{3}$, (*d*) $\$\frac{1}{8}$, (*e*) $\$\frac{3}{8}$, (*f*) 75¢, (*g*) $16\frac{2}{3}$¢, (*h*) $87\frac{1}{2}$¢, etc.

III. At (*a*) 50¢ each, (*b*) 25¢ each, (*c*) $33\frac{1}{3}$¢ each, etc., find the cost of (*d*) 24, (*e*) 36, (*f*) 48, (*g*) 96, etc., articles.

IV. Find the number of articles at (*a*) 50¢ each, (*b*) 25¢ each, (*c*) $33\frac{1}{3}$¢ each, etc., that can be bought for (*d*) \$12, (*e*) \$20, (*f*) \$32, etc.

V. (*a*) $\$9 \div \$\frac{1}{4}$, (*b*) $\$12 \div \$\frac{3}{4}$, (*c*) $\$20 \div \$\frac{2}{3}$, (*d*) \$8 ÷ 25¢, (*e*) \$9 ÷ $33\frac{1}{3}$¢, (*f*) $\$12\frac{1}{2}$ ÷ 50¢.

Advantage should be taken of the short method pursued in finding the cost of 84 articles at 25 cents each, to find the cost of 25 articles at 84 cents each. If the teacher thinks it advisable to require the analysis of problems of this type, it may take the following form:

"At 84 cents each, 100 articles would cost \$84; one-fourth of 100 articles would cost, therefore, one-fourth of \$84, or \$21."

Oral examples of type IV are generally worked by multiplication. A pupil finding the number of 25-cent articles that can be bought for \$20 obtains it by multiplying 20 by 4. Sight exercises such as those given in type V should be employed to show that this kind of example is one in division. When it is written in the form in which it is stated, the operation indicated is division, whether the numbers are written as dollars or as cents. In (*a*) the pupil thinks 25 cents as contained 4 times in 1 dollar; in 20 times 1 dollar, it is, therefore, contained 20 times 4 times, or 80 times. In (*b*) and (*c*), the divisor and the dividend being like numbers, the operation is direct division in each case.

$$\begin{array}{r} (a)\ \$.25\underline{)\$20} \\ 80 \end{array}$$

$$\begin{array}{r} (b)\ 25¢\underline{)\ 2000¢} \\ 80 \end{array}$$

$$\begin{array}{r} (c)\ \$\tfrac{1}{4}\underline{)\$20} \\ 80 \end{array}$$

CHAPTER II

SUBORDINATE OPERATIONS

THE systematic treatment of fractions requires that the pupils should be provided with the means of ascertaining the common denominator of several fractions and of expressing a fraction in its lowest terms. As a preliminary, the average text-book devotes a section to properties of numbers, prime numbers, common factors, least common multiple, greatest common divisor, cancellation, etc.

Pupils develop more interest in these topics when they understand the necessity for them. For this reason, many intelligent teachers take them up incidentally, as occasion arises, instead of following the order of the book.

When pupils are accustomed to indicate operations before performing the work, a problem involving both multiplication and division may be employed to show the application of cancellation; such as the following:

(*a*) How many bushels of wheat can be raised on 63 acres of land at the rate of 400 bushels to 21 acres?

(*a*) $\dfrac{400\text{ bu.} \times \overset{3}{\cancel{63}}}{\cancel{21}} = 1200\text{ bu.}$

(*b*) $\dfrac{\overset{20}{\cancel{400}}\text{ bu.} \times 43}{\cancel{20}} = 860\text{ bu.}$

(*b*) When 400 bushels are raised on 20 acres, how many can be raised on 43 acres at the same rate?

(*c*) When 364 bushels are raised on 20 acres, what should be the yield of 45 acres?

(*c*) $\dfrac{\overset{91}{\cancel{364}}\text{ bu.} \times \overset{9}{\cancel{45}}}{\underset{\cancel{4}}{\cancel{20}}} = 819\text{ bu.}$

In each of the foregoing examples the number of bushels obtained from 1 acre is indicated by writing the given yield as a dividend and the number of acres as the divisor. Without finding the quotient, the product of the latter by the number of acres in the other field is indicated.

In (*a*) the pupil sees that the required yield is 3 times the given yield since the field on which it is produced is 3 times as large as the other, and he can appreciate the fact that this result is indicated by the numbers remaining after the cancellation is performed. In (*b*) the cancellation gives the yield per acre in one field, the product of which by the number of acres in the other field is indicated. In (*c*) the pupils are led to see that the number of acres in the larger field is not an exact multiple of the other number, and also that the number of bushels obtained per acre is not an integer. One pupil may be permitted to work the example by finding the yield per acre, $18\frac{1}{5}$ bu., and multiplying this by the number of acres, 45; another may be led to see that the quantity is $2\frac{1}{4}$ times 364 bushels, since 45 acres is $2\frac{1}{4}$ times 20 acres; the agreement of these two results with the one obtained by cancellation will show the correctness of this method. The fact that this operation rejects equal factors from a divisor and a dividend need not be unnecessarily dwelt upon at this stage.

In connection with this work the divisibility of a number by 2, by 3, by 5, by 9 may receive incidental treatment, which may also be developed in the reduction of some fractions to lowest terms.

The method of finding the least common multiple should not be taught until it is reached in fraction work, when an example in addition of fractions should be given in which the common denominator is not readily determined by inspection.

$$\begin{array}{r} 4\frac{5}{6} \\ 10\frac{13}{15} \\ \frac{17}{24} \\ 8\frac{23}{36} \\ \hline \end{array}$$

$$\begin{array}{r|rrrr} 2 & \cancel{6} & \cancel{15} & 24 & 30 \\ \hline 3 & & & 12 & 15 \\ \hline & & & 4 & 5 \end{array}$$

$$2 \times 3 \times 4 \times 5 = 120$$

The pupil is told to write the denominators in a line, and to cancel any that are factors of any others. He can readily be made to see that a denominator that is divisible by 30 is also divisible by 6 and by 15, and that these two may be omitted from consideration, the question then being the determination of the common denominator of 24 and 30. Even if the pupil can now give it at sight, he should be told to divide the remaining denominators by the smallest number that will divide both, then to divide the quotients in the same way, and finally to obtain the continued product of the two divisors and the final quotients. Using the least common denominator thus obtained, the addition example may then be worked.

There is no immediate necessity for laying stress on the distinction between the expressions, least common multiple and least common denominator. The addition work requires the use of a common denominator, and for convenience this should be as small a number as possible. If the common denominator employed by a pupil is not the smallest possible one, no harm is done, especially when the fraction in the result is reduced to lowest terms.

$$\begin{array}{r} 2\frac{3}{5} \\ 15\frac{5}{8} \\ \frac{7}{10} \\ 4\frac{11}{24} \\ \underline{9\frac{8}{15}} \end{array} \qquad \begin{array}{r} 2\underline{)\cancel{5}\quad \cancel{8}\quad 10\quad 24\quad 15} \\ 3\underline{)\ \cancel{5}\quad 12\quad 15} \\ 4\quad\ \ 5 \\ 2\times 3\times 4\times 5 = 120 \end{array}$$

The accompanying example, or a similar one, may be employed to show that 15, not being divisible by 2, is brought down, also that 5 in the second line is rejected, since it is a factor of 15.

The reasons underlying this method should not be given at this stage, the fact that the number thus obtained is divisible by each denominator being sufficient for the average pupil. In the reviews of the next two years, the more scientific method may be presented and the principles developed.

Instruction in finding the greatest common divisor of two numbers may well be postponed until a pupil's inability to realize that a fraction in his result is not expressed in lowest terms supplies a reason for introducing it. To avoid confusing the pupils by a method for finding the greatest common divisor that bears any resemblance to the one employed in ascertaining the least common multiple, the following is recommended:

Assuming the fraction in a result given by a pupil to be $\frac{87}{120}$, he may be sent to the blackboard to ascertain the largest number that will exactly divide both terms. He is directed to divide the greater by the less, and to use each remainder to divide the previous divisor, etc. Other pupils may then work on the blackboard examples taken from their text-books. It should, however, be impressed upon them that the greatest common divisor should be obtained in this way only when it is necessary. Opportunity should be taken to show that when a numerator is a prime number, a fraction cannot be reduced unless the denominator exactly contains it, etc.

```
87)120(1
    87
    33)87(2
       66
       21)33(1
          21
          12)21(1
             12
              9)12(1
                 9
                 3)9(3
                   9
```

The method by factoring is unnecessary at this time, as pupils able to factor the two numbers comprising the terms of the fraction can readily use these factors in reducing it.

The technical words pertaining to these subordinate operations should be employed by the teachers and pupils as occasion requires, their meaning being obtained by the reading and discussion of the text-book.

CHAPTER III

FRACTIONS

The formal study of fractions in the fifth year should be preceded by the continuation and extension of the primary work. The first examples in addition of mixed numbers should contain only such fractions as have a common denominator readily determined by inspection, and the number of addends should be limited for some time to three or four.

Incidentally, the pupils should hear and use the following expressions, without, however, being required to memorize any formal definitions:

Fraction, numerator, denominator, proper fraction, improper fraction, integer, mixed number, reduction, lowest terms, common denominator, least common multiple, least common denominator, factor, multiple, common divisor, greatest common divisor, prime number, cancellation.

Each of these should be employed by the teacher as the need arises. The fact that in a text-book they are brought together at the beginning of the chapter on fractions, or elsewhere, is no reason for teaching all of them at the same time. The text-book arrangement is a convenient one for purposes of reference; but a pupil should hear a new term only when it is required in connection with the work in hand.

The fraction exercises of the fifth year will depend upon those already taught in the lower grades. As a rule pupils have during the third and fourth years added two or more mixed numbers in which the fractions have had the same denominator, such as:

(*a*) $13\frac{1}{2} + 43\frac{1}{2} + 24$ (*c*) $86 + 2\frac{2}{3} + \frac{1}{3}$
(*b*) $26\frac{1}{4} + 2\frac{3}{4} + 16\frac{1}{4}$ (*d*) $20\frac{1}{5} + 3\frac{2}{5} + 30\frac{3}{5}$

The next step includes such fractions as halves and fourths, halves and sixths, thirds and sixths, the denominator of one fraction in each example being a multiple of the other denominators; for instance:

(*a*) $34\frac{1}{2} + 27\frac{3}{4} + 6\frac{1}{2}$ (*c*) $8\frac{1}{2} + 27\frac{5}{6} + \frac{1}{6}$
(*b*) $16\frac{1}{3} + 9\frac{5}{6} + 30\frac{2}{3}$ (*d*) $16\frac{1}{2} + 4\frac{1}{3} + 72\frac{1}{6}$

The third step contains denominators prime to each other; such as those in the following examples:

(*a*) $6\frac{1}{2} + 4\frac{1}{3}$ (*d*) $39\frac{1}{3} + 13\frac{1}{5}$ (*g*) $37\frac{3}{4} + 19\frac{3}{5}$
(*b*) $3\frac{1}{2} + 2\frac{1}{5}$ (*e*) $62\frac{1}{2} + 15\frac{2}{3}$ (*h*) $29\frac{2}{3} + 16\frac{4}{5}$
(*c*) $8\frac{1}{4} + 5\frac{1}{5}$ (*f*) $23\frac{1}{2} + 18\frac{2}{5}$ (*i*) $18\frac{3}{4} + 20\frac{2}{3}$

If any of these steps have been omitted in the lower grades, the examples in the fifth year should commence at that point.

Assuming that all of the foregoing have been already taught, the first lesson in fractions may well be one in which a pupil is told to write on the board and add the following, for instance:

$$\begin{array}{r|r} & 36 \\ \hline 27\frac{3}{4} & 27 \\ 8\frac{1}{6} & 6 \\ 26\frac{5}{9} & 20 \\ \frac{7}{12} & 21 \\ \hline \textit{Ans.}\ 63\frac{1}{18} & \frac{74}{36} = 2\frac{2}{36} = 2\frac{1}{18} \end{array}$$

He is first asked for the common denominator, an expression already used. If it has not been employed heretofore, he may be asked for a number that will exactly contain the denominators, 4, 6, 9, 12. If he should say 72, he is told that 72 is a common denominator, but that the smallest common denominator is desired. When this is obtained from him or from some other pupil, he is told to write it in the usual way, and to reduce the several fractions, writing only the numerators. The sum of the new numerators is then obtained and written over the common denominator. The improper fraction $\frac{74}{36}$ is reduced to the mixed number $2\frac{2}{36}$, and the latter is again written with its fraction expressed in lowest terms, $2\frac{1}{18}$. The fraction $\frac{1}{18}$ is now written under the fractions in the original example, and 2 is carried to the integers, the result being $63\frac{1}{18}$.

The pupil should not use any superfluous figures. The reduction of $\frac{74}{36}$ does not require that these numbers be written again.

Other pupils work on the board examples that may be even simpler than the foregoing, containing in some instances only two addends; such as:

$$\begin{array}{rrrrrrrr} (a) & 27\frac{3}{4} & (b) & 63\frac{4}{9} & (c) & 30\frac{5}{8} & (d) & 44\frac{3}{8} \\ & \underline{+\ 8\frac{5}{6}} & & \underline{+\ 18\frac{5}{12}} & & \underline{+\ 6\frac{7}{12}} & & \underline{+\ 7\frac{5}{6}} \end{array}$$

In these, the denominators are not prime to each other and the least common denominator will be less than their product.

When three addends are used the denominator of the new one may be a factor of one of the other denominators.

$$\begin{array}{rrrrrrrr} (a) & 27\frac{3}{4} & (b) & 27\frac{3}{4} & (c) & 63\frac{4}{9} & (d) & 63\frac{4}{9} \\ & 8\frac{5}{6} & & 8\frac{5}{6} & & 18\frac{5}{12} & & 18\frac{5}{12} \\ & \underline{3\frac{1}{3}} & & \underline{20\frac{1}{2}} & & \underline{3\frac{2}{3}} & & \underline{3\frac{1}{6}} \end{array}$$

$$\begin{array}{rrrrrrrr} (e) & 63\frac{4}{9} & (f) & 63\frac{4}{9} & (g) & 30\frac{5}{8} & (h) & 30\frac{5}{8} \\ & 18\frac{5}{12} & & 18\frac{5}{12} & & 6\frac{7}{12} & & 6\frac{7}{12} \\ & \underline{2\frac{3}{4}} & & \underline{7\frac{1}{2}} & & \underline{20\frac{2}{3}} & & \underline{20\frac{1}{2}} \end{array}$$

Examples of this kind show the pupils that in ascertaining a common denominator they omit from consideration a denominator that is a factor of any other; that, for instance in finding the least common denominator of 4, 6, and 3, in (*a*), the last may be rejected, as it is a factor of 6, and that only 4 and 6 need be considered.

The denominators should generally be small numbers, and confined largely to the so-called business fractions.

The drill exercises of the fifth year may include all of the types heretofore employed. These are particularly needed after the summer vacation. They may be answered from the blackboard or from cards.

The simplest set contains such combinations as do not require the reduction of the sums:

$$\begin{array}{llllllll} \text{I.} & (a)\ \ \tfrac{1}{5} & (b)\ \ \tfrac{3}{5} & (c)\ \ \tfrac{2}{5} & (d)\ \ \tfrac{1}{9} & (e)\ \ \tfrac{1}{3} & (f)\ \ \tfrac{2}{9} & (g)\ \ \tfrac{4}{9} \\ & \quad \underline{+\tfrac{2}{5}} & \quad \underline{+\tfrac{1}{5}} & \quad \underline{+\tfrac{2}{5}} & \quad \underline{+\tfrac{4}{9}} & \quad \underline{+\tfrac{1}{3}} & \quad \underline{+\tfrac{2}{9}} & \quad \underline{+\tfrac{4}{9}} \end{array}$$

In the following the sums are reduced to mixed numbers or to integers:

$$\begin{array}{llllllll} \text{II.} & (a)\ \ \tfrac{2}{5} & (b)\ \ \tfrac{3}{4} & (c)\ \ \tfrac{4}{9} & (d)\ \ \tfrac{3}{5} & (e)\ \ \tfrac{5}{8} & (f)\ \ \tfrac{4}{7} & (g)\ \ \tfrac{2}{3} \\ & \quad \underline{+\tfrac{4}{5}} & \quad \underline{+\tfrac{1}{4}} & \quad \underline{+\tfrac{7}{9}} & \quad \underline{+\tfrac{3}{5}} & \quad \underline{+\tfrac{3}{8}} & \quad \underline{+\tfrac{6}{7}} & \quad \underline{+\tfrac{2}{3}} \end{array}$$

In the following, the sums are reduced to lowest terms:

$$\begin{array}{llllllll} \text{III.} & (a)\ \ \tfrac{3}{8} & (b)\ \ \tfrac{2}{9} & (c)\ \ \tfrac{3}{10} & (d)\ \ \tfrac{1}{6} & (e)\ \ \tfrac{5}{12} & (f)\ \ \tfrac{1}{9} & (g)\ \ \tfrac{1}{8} \\ & \quad \underline{+\tfrac{3}{8}} & \quad \underline{+\tfrac{4}{9}} & \quad \underline{+\tfrac{1}{10}} & \quad \underline{+\tfrac{1}{6}} & \quad \underline{+\tfrac{5}{12}} & \quad \underline{+\tfrac{2}{9}} & \quad \underline{+\tfrac{5}{8}} \end{array}$$

In the following the improper fractions in the results are changed to mixed numbers, and the fractions are given in lowest terms:

$$\begin{array}{llllllll} \text{IV.} & (a)\ \ \tfrac{5}{8} & (b)\ \ \tfrac{5}{9} & (c)\ \ \tfrac{9}{10} & (d)\ \ \tfrac{5}{6} & (e)\ \ \tfrac{3}{4} & (f)\ \ \tfrac{7}{12} & (g)\ \ \tfrac{7}{8} \\ & \quad \underline{+\tfrac{5}{8}} & \quad \underline{+\tfrac{7}{9}} & \quad \underline{+\tfrac{3}{10}} & \quad \underline{+\tfrac{5}{6}} & \quad \underline{+\tfrac{3}{4}} & \quad \underline{+\tfrac{7}{12}} & \quad \underline{+\tfrac{5}{8}} \end{array}$$

The fractions in the next four sets are similar, respectively, to those in the preceding combinations, but one addend is a mixed number.

$$\text{V.}\ (a)\ \begin{array}{r}5\frac{1}{5}\\ +\ \frac{2}{5}\\ \hline\end{array}\quad (b)\ \begin{array}{r}\frac{1}{5}\\ +\ 7\frac{3}{5}\\ \hline\end{array}\quad (c)\ \begin{array}{r}4\frac{2}{5}\\ +\ \frac{2}{5}\\ \hline\end{array}\quad (d)\ \begin{array}{r}\frac{1}{3}\\ +\ 6\frac{1}{3}\\ \hline\end{array}\quad (e)\ \begin{array}{r}8\frac{2}{9}\\ +\ \frac{2}{9}\\ \hline\end{array}$$

$$\text{VI.}\ (a)\ \begin{array}{r}\frac{2}{5}\\ +\ 3\frac{4}{5}\\ \hline\end{array}\quad (b)\ \begin{array}{r}4\frac{3}{4}\\ +\ \frac{1}{4}\\ \hline\end{array}\quad (c)\ \begin{array}{r}\frac{7}{9}\\ +\ 5\frac{4}{9}\\ \hline\end{array}\quad (d)\ \begin{array}{r}7\frac{3}{5}\\ +\ \frac{3}{5}\\ \hline\end{array}\quad (e)\ \begin{array}{r}\frac{2}{3}\\ +\ 8\frac{2}{3}\\ \hline\end{array}$$

$$\text{VII.}\ (a)\ \begin{array}{r}5\frac{3}{8}\\ +\ \frac{3}{8}\\ \hline\end{array}\quad (b)\ \begin{array}{r}\frac{4}{9}\\ +\ 6\frac{2}{9}\\ \hline\end{array}\quad (c)\ \begin{array}{r}4\frac{3}{10}\\ +\ \frac{1}{10}\\ \hline\end{array}\quad (d)\ \begin{array}{r}\frac{1}{6}\\ +\ 7\frac{1}{6}\\ \hline\end{array}\quad (e)\ \begin{array}{r}8\frac{5}{12}\\ +\ \frac{5}{12}\\ \hline\end{array}$$

$$\text{VIII.}\ (a)\ \begin{array}{r}\frac{5}{8}\\ +\ 4\frac{5}{8}\\ \hline\end{array}\quad (b)\ \begin{array}{r}3\frac{5}{9}\\ +\ \frac{7}{9}\\ \hline\end{array}\quad (c)\ \begin{array}{r}\frac{3}{10}\\ +\ 6\frac{9}{10}\\ \hline\end{array}\quad (d)\ \begin{array}{r}5\frac{5}{6}\\ +\ \frac{5}{6}\\ \hline\end{array}\quad (e)\ \begin{array}{r}\frac{3}{4}\\ +\ 7\frac{3}{4}\\ \hline\end{array}$$

In the following four sets, both addends are mixed numbers, the fractions, however, being similar, respectively, to those in the preceding sets:

$$\text{IX.}\ (a)\ \begin{array}{r}5\frac{1}{5}\\ +\ 2\frac{2}{5}\\ \hline\end{array}\quad (b)\ \begin{array}{r}7\frac{3}{5}\\ +\ 1\frac{1}{5}\\ \hline\end{array}\quad (c)\ \begin{array}{r}4\frac{2}{5}\\ +\ 3\frac{2}{5}\\ \hline\end{array}\quad (d)\ \begin{array}{r}6\frac{1}{3}\\ +\ 2\frac{1}{3}\\ \hline\end{array}\quad (e)\ \begin{array}{r}3\frac{2}{9}\\ +\ 5\frac{2}{9}\\ \hline\end{array}$$

$$\text{X.}\ (a)\ \begin{array}{r}5\frac{2}{5}\\ +\ 3\frac{4}{5}\\ \hline\end{array}\quad (b)\ \begin{array}{r}4\frac{3}{4}\\ +\ 4\frac{1}{4}\\ \hline\end{array}\quad (c)\ \begin{array}{r}2\frac{7}{9}\\ +\ 5\frac{4}{9}\\ \hline\end{array}\quad (d)\ \begin{array}{r}7\frac{3}{5}\\ +\ 1\frac{3}{5}\\ \hline\end{array}\quad (e)\ \begin{array}{r}3\frac{2}{3}\\ +\ 3\frac{2}{3}\\ \hline\end{array}$$

$$\text{XI.}\ (a)\ \begin{array}{r}5\frac{3}{8}\\ +\ 1\frac{3}{8}\\ \hline\end{array}\quad (b)\ \begin{array}{r}2\frac{4}{9}\\ +\ 6\frac{2}{9}\\ \hline\end{array}\quad (c)\ \begin{array}{r}4\frac{3}{10}\\ +\ 3\frac{1}{10}\\ \hline\end{array}\quad (d)\ \begin{array}{r}1\frac{1}{6}\\ +\ 7\frac{1}{6}\\ \hline\end{array}\quad (e)\ \begin{array}{r}3\frac{5}{12}\\ +\ 4\frac{5}{12}\\ \hline\end{array}$$

$$\text{XII.}\ (a)\ \begin{array}{r}5\frac{5}{8}\\ +\ 1\frac{5}{8}\\ \hline\end{array}\quad (b)\ \begin{array}{r}3\frac{5}{9}\\ +\ 4\frac{7}{9}\\ \hline\end{array}\quad (c)\ \begin{array}{r}1\frac{3}{10}\\ +\ 6\frac{9}{10}\\ \hline\end{array}\quad (d)\ \begin{array}{r}5\frac{5}{6}\\ +\ 2\frac{5}{6}\\ \hline\end{array}\quad (e)\ \begin{array}{r}2\frac{3}{4}\\ +\ 3\frac{3}{4}\\ \hline\end{array}$$

In classes taking up fractions for the first time, these drills are preceded by the ones recommended for the primary grades, the first set being limited to those containing halves alone, the next set to those containing fourths alone, and the next to those containing thirds alone.

Some teachers may prefer a different order for the twelve suggested sets, following I by V and IX, II by VI and X, etc.; thus:

$$\begin{array}{lrlrlr}
\text{I.} & \frac{1}{5}\text{ etc.} & \text{V.} & 5\frac{1}{5}\text{ etc.} & \text{IX.} & 5\frac{1}{5}\text{ etc.} \\
 & \underline{+\ \frac{2}{5}\phantom{\text{ etc.}}} & & \underline{+\ \ \frac{2}{5}\phantom{\text{ etc.}}} & & \underline{+\ 2\frac{2}{5}\phantom{\text{ etc.}}} \\
\text{II.} & \frac{2}{5}\text{ etc.} & \text{VI.} & \frac{2}{5}\text{ etc.} & \text{X.} & 5\frac{2}{5}\text{ etc.} \\
 & \underline{+\ \frac{4}{5}\phantom{\text{ etc.}}} & & \underline{+\ 3\frac{4}{5}\phantom{\text{ etc.}}} & & \underline{+\ 3\frac{4}{5}\phantom{\text{ etc.}}} \\
\text{III.} & \frac{3}{8}\text{ etc.} & \text{VII.} & 5\frac{3}{8}\text{ etc.} & \text{XI.} & 6\frac{3}{8}\text{ etc.} \\
 & \underline{+\ \frac{3}{8}\phantom{\text{ etc.}}} & & \underline{+\ \ \frac{3}{8}\phantom{\text{ etc.}}} & & \underline{+\ 1\frac{3}{8}\phantom{\text{ etc.}}} \\
\text{IV.} & \frac{5}{8}\text{ etc.} & \text{VIII.} & 5\frac{5}{8}\text{ etc.} & \text{XII.} & 5\frac{5}{8}\text{ etc.} \\
 & \underline{+\ \frac{5}{8}\phantom{\text{ etc.}}} & & \underline{+\ \ \frac{5}{8}\phantom{\text{ etc.}}} & & \underline{+\ 1\frac{5}{8}\phantom{\text{ etc.}}}
\end{array}$$

In order that pupils may be accustomed to dispense with unnecessary side calculations, blackboard exercises should be given in which the pupils write only the answers. The number of addends will depend upon the fractions employed; four being the limit when the fractions have the same denominator; three, when the denominator of one of the fractions is the common denominator; and two, in other cases; as, for example:

$$\begin{array}{llrlrlrlr}
\text{I.} & (a) & 13\frac{1}{5} & (b) & 20\frac{2}{9} & (c) & 3\frac{1}{8} & (d) & 64\frac{1}{10} \\
 & & 4\frac{2}{5} & & 3\frac{5}{9} & & 22\frac{3}{8} & & 10\frac{3}{10} \\
 & & 25\frac{3}{5} & & 6\frac{7}{9} & & \frac{5}{8} & & 7\frac{7}{10} \\
 & & \underline{6\frac{4}{5}} & & \underline{\frac{1}{9}} & & \underline{16\frac{7}{8}} & & \underline{2\frac{9}{10}} \\
\text{II.} & (a) & 13\frac{1}{2} & (b) & 20\frac{1}{2} & (c) & 3\frac{1}{2} & (d) & 64\frac{2}{3} \\
 & & 4\frac{2}{3} & & 34\frac{3}{4} & & 22\frac{3}{5} & & 10\frac{3}{4} \\
 & & \underline{25\frac{5}{6}} & & \underline{6\frac{3}{8}} & & \underline{45\frac{7}{10}} & & \underline{7\frac{1}{12}} \\
\text{III.} & (a) & 13\frac{3}{8} & (b) & 20\frac{2}{9} & (c) & 3\frac{3}{4} & (d) & 64\frac{5}{6} \\
 & & \underline{4\frac{5}{12}} & & \underline{34\frac{1}{12}} & & \underline{22\frac{7}{10}} & & \underline{10\frac{1}{9}}
\end{array}$$

Oral problems should contain only such mixed numbers as do not present too many difficulties. As a rule they should have only two addends, the fractions being similar to those given in the sight types.

In adding $\frac{5}{8} + \frac{5}{8}$, the pupil will first think $\frac{10}{8}$, then $1\frac{2}{8}$, then $1\frac{1}{4}$. In adding $4\frac{5}{8} + \frac{5}{8}$, a similar procedure gives the successive results as $4\frac{10}{8}$, $5\frac{2}{8}$, $5\frac{1}{4}$. When both addends are mixed numbers, such as $5\frac{5}{8} + 1\frac{5}{8}$, the first addend is increased by the integer of the second, and then by the fraction; as follows:

$$5\tfrac{5}{8} + 1\tfrac{5}{8} = 5\tfrac{5}{8} + 1\ (6\tfrac{5}{8}) + \tfrac{5}{8}\ (6\tfrac{10}{8}, 7\tfrac{2}{8}, 7\tfrac{1}{4})$$

Although the greater portion of the time assigned to addition of fractions should be given to those whose common denominators are determinable by inspection, examination papers of various kinds require that the pupils should be made acquainted with the procedure to be followed with other fractions.

The first example of this kind gives an opportunity for the introduction of the least common multiple, which is discussed in the last chapter.

In adding such fractions as are contained in the accompanying example, another extra column may be used to contain the quotient of the common denominator by each given denominator. Thus 135 divided by 15 gives 9, and this number is written in the first column, as are also the respective quotients of 135 divided by 27, 9, 3, and 5. Each of these is multiplied by the corresponding numerator of the given fractions: 9 × 13, 5 × 19, 15 × 5, 45 × 2, and 27 × 4; and the products

	135	
$15\frac{13}{15}$	9	117
$20\frac{19}{27}$	5	95
$8\frac{5}{9}$	15	75
$\frac{2}{3}$	45	90
$16\frac{4}{5}$	27	108
$62\frac{16}{27}$ *Ans.*		$\frac{485}{135} = 3\frac{80}{135} = 3\frac{16}{27}$

117, 95, 75, 90, and 108 are written in the next column. This arrangement supplies all the numbers needed to reduce the given fractions to equivalent ones having 135 for their denominator, and pupils should be expected to use no others. The fraction $\frac{80}{135}$ is reduced by dividing both terms by 5. There is no need as yet for presenting a method of obtaining the greatest common divisor.

$13\frac{3}{8}$
27
$5\frac{4}{9}$
$\underline{15}\frac{5}{9}$

An occasional example may be given in the review exercises of the upper grades similar to the one herewith. In this, the sum of two of the fractions being 1, it is unnecessary to employ the common denominator.

In the written problems of this stage the number of addends should be fewer than in the abstract exercises, two, as a rule, being sufficient, especially when the problem involves other operations. All of this work in addition of fractions should be done without setting aside any portion of the time for formal lessons in reduction. When fractions in the addends are changed to equivalent ones with a common denominator, pupils are learning to reduce to higher terms, but they need not be informed of this fact just yet. In changing an improper fraction in the sum to a mixed number, they are employing another form of reduction; and in expressing the fraction in the final result in lowest terms they are learning still another kind; but the discussion of reduction in its different types may well be postponed until the last two years of the course, when the various topics are classified and studied from the standpoint of the science of arithmetic.

Although not specified as reduction exercises, numerous miscellaneous examples should be employed in the sight and oral work which will give the necessary facility, as well as afford practice in number combinations. The text-books supply many examples, which the teacher should supplement by others of his own. They can be used as drills in the various forms suggested for exercises of this kind in the lower grades. When written on the blackboard, the teacher places a pointer on each and calls for rapid replies. The following are types:

I. (*a*) Reduce to 24ths

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$

(*b*) Reduce to 30ths

$$\begin{array}{ccccccc} \frac{1}{2}, & \frac{1}{3}, & \frac{1}{5}, & \frac{1}{6}, & \frac{1}{10}, & \frac{1}{15}, & \frac{2}{3}, \\ \frac{2}{5}, & \frac{3}{5}, & \frac{4}{5}, & \frac{5}{6}, & \frac{3}{10}, & \frac{7}{10}, & \frac{9}{10}, \\ \frac{2}{15}, & \frac{4}{15}, & \frac{7}{15}, & \frac{8}{15}, & \frac{11}{15}, & \frac{13}{15}, & \frac{14}{15}, \end{array}$$

etc.

II. Reduce to lowest terms

$$\begin{array}{cccccc} \frac{15}{45}, & \frac{16}{64}, & \frac{17}{51}, & \frac{28}{84}, & \frac{23}{69}, & \frac{32}{96}, \\ \frac{30}{45}, & \frac{48}{64}, & \frac{34}{51}, & \frac{56}{84}, & \frac{46}{69}, & \frac{64}{96}, \end{array}$$

etc.

This type is valuable in giving the pupils an acquaintance with multiples and factors that is needed in cancellation work.

III. Reduce to mixed numbers

$\frac{49}{15}, \frac{51}{16}, \frac{35}{17}, \frac{55}{18}, \frac{40}{19}, \frac{65}{21},$

etc.

The denominators should be greater than 13, to give practice in handling large numbers; but the numerators should be such as to make the exercises simple enough for this stage. This requires that the remainders be not too large at first.

Drills of these types are more interesting than similar combinations presented as exercises with whole numbers, which the pupils are apt to consider as belonging in primary grades.

As an introduction to subtraction of fractions, exercises similar to the following may be written on the blackboard for sight solution. They are the reverse of some of those given for addition.

$$\text{I.}\ (a)\ \begin{array}{r}\frac{3}{5}\\ +\ ?\\ \hline 1\end{array}\qquad (b)\ \begin{array}{r}\frac{1}{3}\\ +\ ?\\ \hline 1\end{array}\qquad (c)\ \begin{array}{r}\frac{2}{9}\\ +\ ?\\ \hline 1\end{array}\qquad (d)\ \begin{array}{r}\frac{2}{5}\\ +\ ?\\ \hline 1\end{array}\qquad (e)\ \begin{array}{r}\frac{4}{9}\\ +\ ?\\ \hline 1\end{array}\qquad (f)\ \begin{array}{r}\frac{4}{5}\\ +\ ?\\ \hline 1\end{array}$$

$$\text{II.}\ (a)\ \begin{array}{r}?\\ +\ \frac{3}{5}\\ \hline 1\frac{1}{5}\end{array}\qquad (b)\ \begin{array}{r}?\\ +\ \frac{2}{3}\\ \hline 1\frac{1}{3}\end{array}\qquad (c)\ \begin{array}{r}?\\ +\ \frac{2}{9}\\ \hline 1\frac{1}{9}\end{array}\qquad (d)\ \begin{array}{r}?\\ +\ \frac{2}{5}\\ \hline 1\frac{1}{5}\end{array}\qquad (e)\ \begin{array}{r}?\\ +\ \frac{4}{9}\\ \hline 1\frac{2}{9}\end{array}\qquad (f)\ \begin{array}{r}?\\ +\ \frac{4}{5}\\ \hline 1\frac{3}{5}\end{array}$$

The regular sight exercises in subtraction of fractions follow the order pursued in addition, the sum in each of the latter being the minuend in a subtraction exercise, and either of the two addends being taken as the subtrahend. The following include the foregoing types and some new ones:

$$\text{I.}\ (a)\ \begin{array}{r}\frac{4}{5}\\ -\ \frac{1}{5}\\ \hline\end{array}\qquad (b)\ \begin{array}{r}\frac{4}{5}\\ -\ \frac{2}{5}\\ \hline\end{array}\qquad (c)\ \begin{array}{r}\frac{4}{5}\\ -\ \frac{3}{5}\\ \hline\end{array}\qquad (d)\ \begin{array}{r}\frac{5}{9}\\ -\ \frac{1}{9}\\ \hline\end{array}\qquad (e)\ \begin{array}{r}\frac{5}{9}\\ -\ \frac{4}{9}\\ \hline\end{array}\qquad (f)\ \begin{array}{r}\frac{2}{3}\\ -\ \frac{1}{3}\\ \hline\end{array}$$

$$\text{II.}\ (a)\ \begin{array}{r}\frac{3}{4}\\ -\ \frac{3}{8}\\ \hline\end{array}\qquad (b)\ \begin{array}{r}\frac{2}{3}\\ -\ \frac{4}{9}\\ \hline\end{array}\qquad (c)\ \begin{array}{r}\frac{2}{5}\\ -\ \frac{1}{10}\\ \hline\end{array}\qquad (d)\ \begin{array}{r}\frac{1}{3}\\ -\ \frac{1}{6}\\ \hline\end{array}\qquad (e)\ \begin{array}{r}\frac{5}{6}\\ -\ \frac{5}{12}\\ \hline\end{array}\qquad (f)\ \begin{array}{r}\frac{1}{3}\\ -\ \frac{1}{9}\\ \hline\end{array}$$

$$\text{III.}\ (a)\ \begin{array}{r}1\\ -\ \frac{1}{8}\\ \hline\end{array}\qquad (b)\ \begin{array}{r}2\\ -\ \frac{2}{7}\\ \hline\end{array}\qquad (c)\ \begin{array}{r}3\\ -\ \frac{5}{6}\\ \hline\end{array}\qquad (d)\ \begin{array}{r}4\\ -\ \frac{3}{5}\\ \hline\end{array}\qquad (e)\ \begin{array}{r}5\\ -\ \frac{1}{4}\\ \hline\end{array}\qquad (f)\ \begin{array}{r}6\\ -\ \frac{2}{3}\\ \hline\end{array}$$

$$\text{IV.}\ (a)\ \begin{array}{r}1\frac{1}{5}\\ -\ \frac{3}{5}\\ \hline\end{array}\qquad (b)\ \begin{array}{r}1\frac{1}{9}\\ -\ \frac{5}{9}\\ \hline\end{array}\qquad (c)\ \begin{array}{r}1\frac{2}{9}\\ -\ \frac{4}{9}\\ \hline\end{array}\qquad (d)\ \begin{array}{r}1\frac{2}{5}\\ -\ \frac{4}{5}\\ \hline\end{array}\qquad (e)\ \begin{array}{r}1\frac{1}{3}\\ -\ \frac{2}{3}\\ \hline\end{array}\qquad (f)\ \begin{array}{r}1\frac{5}{9}\\ -\ \frac{7}{9}\\ \hline\end{array}$$

$$\text{V.}\ (a)\ \begin{array}{r}1\frac{3}{4}\\ -\ \frac{3}{8}\\ \hline\end{array}\qquad (b)\ \begin{array}{r}1\frac{2}{3}\\ -\ \frac{2}{9}\\ \hline\end{array}\qquad (c)\ \begin{array}{r}1\frac{3}{5}\\ -\ \frac{1}{10}\\ \hline\end{array}\qquad (d)\ \begin{array}{r}1\frac{5}{6}\\ -\ \frac{2}{3}\\ \hline\end{array}\qquad (e)\ \begin{array}{r}1\frac{7}{8}\\ -\ \frac{1}{4}\\ \hline\end{array}\qquad (f)\ \begin{array}{r}1\frac{5}{8}\\ -\ \frac{1}{2}\\ \hline\end{array}$$

$$\text{VI.}\ (a)\ \begin{array}{r}1\frac{1}{4}\\ -\ \frac{5}{8}\\ \hline\end{array}\qquad (b)\ \begin{array}{r}1\frac{1}{3}\\ -\ \frac{4}{9}\\ \hline\end{array}\qquad (c)\ \begin{array}{r}1\frac{1}{5}\\ -\ \frac{7}{10}\\ \hline\end{array}\qquad (d)\ \begin{array}{r}1\frac{1}{6}\\ -\ \frac{2}{3}\\ \hline\end{array}\qquad (e)\ \begin{array}{r}1\frac{1}{8}\\ -\ \frac{3}{4}\\ \hline\end{array}\qquad (f)\ \begin{array}{r}1\frac{1}{9}\\ -\ \frac{1}{3}\\ \hline\end{array}$$

etc.

Before proceeding to the more difficult examples in written subtraction of fractions, the teacher should require the pupils to write only the answers to exercises placed on the blackboard, the types used in the sight exercises being followed, but each term consisting of a mixed number.

$$\begin{array}{r} 46\frac{2}{9} \\ -\,28\frac{4}{9} \\ \hline \end{array}$$

In solving a blackboard exercise of the type given in set IV of the sight drills, the teacher has a choice of two methods. The usual one is to find the difference between the fraction in the subtrahend $\frac{4}{9}$ and that in the minuend increased by 1, making it $\frac{11}{9}$. The difference $\frac{7}{9}$ is then written, 1 is added to the 8 of the subtrahend, or is deducted from 6 of the minuend, etc. When, however, the denominator of the fractions is a large number, the pupil may use a simpler method of diminishing $1\frac{17}{25}$ by $\frac{19}{25}$ than the one usually followed. By the latter, $1\frac{17}{25}$ is changed to $\frac{42}{25}$, from which $\frac{19}{25}$ is taken. The other method consists in first subtracting $\frac{19}{25}$ from 1, and increasing this result by $\frac{17}{25}$. That is,

$$\begin{array}{r} 46\frac{17}{25} \\ -\,28\frac{19}{25} \\ \hline \end{array}$$

$$1 + \tfrac{17}{25} - \tfrac{19}{25} = 1 - \tfrac{19}{25}\,(\tfrac{6}{25}) + \tfrac{17}{25}\,(\tfrac{23}{25})$$

$$\begin{array}{r|l} 46\frac{17}{25} & 42 \\ -\,28\frac{19}{25} & \\ \hline \end{array}$$

If the pupil prefers to use the other form, he may reduce $1\frac{17}{25}$ to $\frac{42}{25}$ by writing the sum of 25 and 17, as is shown in the accompanying example. From 42 the pupil should subtract 19 without rewriting the latter directly under the former.

$$\begin{array}{r|c|cc} & 60 & & \\ \cline{2-2} 46\frac{7}{15} & 28 & 88 & \\ 28\frac{11}{12} & 55 & & \\ \hline \textit{Ans.}\ 17\frac{11}{20} & \frac{33}{60} & = & \frac{11}{20} \end{array}$$

When the fractions have different denominators the common denominator is written above the new numerators, as in addition of fractions. The combination of the new numerator (28) with the common denominator (60) may be written to the right if the pupil finds it necessary. An extra column, as is suggested in addition examples containing large denominators, may be used for the quotients. The last column in the illustration is used for the sum of 126 and 135 unless the pupil takes 130 from 135 (5) and increases this remainder by 126.

$$\begin{array}{r|c|c|l} & \multicolumn{2}{c}{135} & \\ \cline{2-3} 46\frac{14}{15} & 9 & 126 & 261 \\ -\,28\frac{26}{27} & 5 & 130 & \\ \hline 17\frac{131}{135}\ \mathit{Ans.} & \multicolumn{1}{c|}{} & \frac{131}{135} & \end{array}$$

In the multiplication work of the lower grades, only one of the factors, as a rule, is fractional, the types being

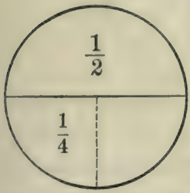
(*a*) $\frac{3}{4} \times 48$ (*b*) $48 \times \frac{3}{4}$ (*c*) $16 \times 3\frac{1}{2}$ (*d*) $16\frac{1}{2} \times 4$

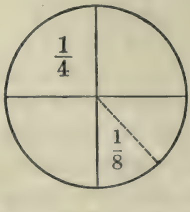
The new work of this grade consists in finding products of two fractional factors.

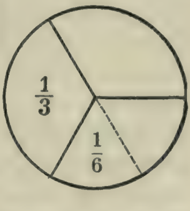
The earliest exercises are such as:

I. (*a*) $\frac{1}{2}$ of $\frac{1}{2}$ (*b*) $\frac{1}{2}$ of $\frac{1}{4}$ (*c*) $\frac{1}{2}$ of $\frac{1}{3}$ (*d*) $\frac{1}{2}$ of $\frac{1}{5}$

At this stage of the fraction work, a beginning should be made towards the development in the child's mind of the meaning of a fraction, but care should be taken not to confuse him by proceeding too rapidly. Although the finding of $\frac{1}{2}$ of $\frac{1}{3}$ involves the division of 1 third into 2 equal parts, these examples are intended to lead up to the multiplication of two mixed numbers, and the fact that multiplication by $\frac{1}{2}$ is the same as division by 2 should not be made too prominent for the present. A pupil may divide a circle into halves, then separate one of the halves into halves, and write in one of the last two parts its name.







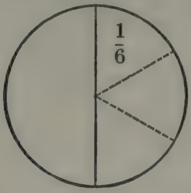
Another follows the same plan in finding $\frac{1}{2}$ of 1 fourth, and still another in finding $\frac{1}{2}$ of 1 third.

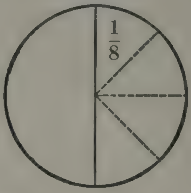
Opportunity is taken to fix the meaning of *denominator*, the term that shows into how many parts a unit is divided; also to develop the fact that the larger the denominator, the smaller the fraction. The number of quarters in 1 half, the number of eighths in 1 half and 1 quarter, respectively, the number of sixths in 1 half and 1 third, respectively, are asked. Without further drawing the pupils should be able to give the answer to $\frac{1}{2}$ of $\frac{1}{5}$, $\frac{1}{2}$ of $\frac{1}{6}$, and to give the reason in each case for their answers.

The next set may include the preceding combinations in this form:

II. (*a*) $\frac{1}{3}$ of $\frac{1}{2}$ (*b*) $\frac{1}{4}$ of $\frac{1}{2}$ (*c*) $\frac{1}{5}$ of $\frac{1}{2}$ (*d*) $\frac{1}{6}$ of $\frac{1}{2}$

The first pupil divides a circle into halves, then divides one of the halves into thirds, and writes in one of the divisions its value. A comparison of the two diagrams shows the pupil that $\frac{1}{2}$ of $\frac{1}{3}$ and $\frac{1}{3}$ of $\frac{1}{2}$ are the same. The fact that $\frac{1}{2}$ of $\frac{1}{4}$ and $\frac{1}{4}$ of $\frac{1}{2}$ are the same is brought out in a like manner. In making these diagrams solid lines are used to separate the first three circles into halves, quarters, and thirds, respectively, a broken line being employed to divide one of the portions into two equal parts. Each of the next two circles is divided into halves by a solid line, broken ones being used to divide one of the halves of the first into three equal parts, and one of the halves of the second into four.





The next type should be preceded by a few exercises placed upon the blackboard to show the meaning of the *numerator* of a fraction. Thus:

$\frac{1}{2}$ of 2 thirds = ? $\frac{1}{3}$ of 3 fourths = ? etc.

The same examples are then written thus:

III. (*a*) $\frac{1}{2}$ of $\frac{2}{3}$ (*b*) $\frac{1}{3}$ of $\frac{3}{4}$ (*c*) $\frac{1}{2}$ of $\frac{4}{5}$ (*d*) $\frac{1}{3}$ of $\frac{6}{7}$

These hardly call for objective illustration, nor do they require analysis. A pupil might, however, be asked how a boy could share 1 third of an apple pie and 1 third of a peach pie with his sister, to bring out the fact that he could do so by halving each third and giving his sister 2 sixths, but that if both thirds were of the same variety, one of the thirds would constitute an equal division of the portions.

The combinations of the previous type may now be presented in this form:

IV. (*a*) $\frac{2}{3}$ of $\frac{1}{2}$ (*b*) $\frac{3}{4}$ of $\frac{1}{3}$ (*c*) $\frac{4}{5}$ of $\frac{1}{2}$ (*d*) $\frac{6}{7}$ of $\frac{1}{3}$

A pupil should explain (*a*) by analyzing it in some such form as this: 1 third of $\frac{1}{2}$ is 1 sixth, therefore 2 thirds of $\frac{1}{2}$ are 2 sixths, or 1 third. If a diagram should be considered necessary, a pupil could use the form employed in obtaining 1 third of $\frac{1}{2}$, and show that 2 of these divisions constitute 1 third of the circle.

To emphasize the fact that one-half of a fraction is obtained by doubling its denominator, etc., the following may be presented:

V. (*a*) $\frac{1}{2}$ of $\frac{3}{4}$ (*b*) $\frac{1}{3}$ of $\frac{2}{3}$ (*c*) $\frac{1}{4}$ of $\frac{3}{4}$ (*d*) $\frac{1}{2}$ of $\frac{3}{5}$

or the same combinations in this form:

VI. (*a*) $\frac{3}{4}$ of $\frac{1}{2}$ (*b*) $\frac{2}{3}$ of $\frac{1}{3}$ (*d*) $\frac{3}{4}$ of $\frac{1}{4}$ (*e*) $\frac{3}{5}$ of $\frac{1}{2}$

In V, the pupil analyzes (*a*) by considering that each of

3 quarters is halved, making the result 3 eighths; in VI, he finds that since 1 fourth of $\frac{1}{2}$ is 1 eighth, 3 fourths of $\frac{1}{2}$ are 3 times 1 eighth, or 3 eighths.

At this point it is advisable to replace the word "of" by the sign of multiplication, the pupils being informed that exercises of this kind are considered as examples in multiplication of fractions. They have already used the multiplication sign in finding $\frac{3}{4}$ of 48, which they have also written $\frac{3}{4} \times 48$ or $48 \times \frac{3}{4}$, so that no explanation is required.

The drill exercises will now consist of the following types, to which the pupils give prompt answers, as they include only the combinations heretofore employed:

I. (*a*) $\frac{1}{2} \times \frac{1}{2}$ (*b*) $\frac{1}{2} \times \frac{1}{4}$ (*c*) $\frac{1}{2} \times \frac{1}{3}$ (*d*) $\frac{1}{2} \times \frac{1}{5}$
II. (*a*) $\frac{1}{3} \times \frac{1}{2}$ (*b*) $\frac{1}{4} \times \frac{1}{2}$ (*c*) $\frac{1}{5} \times \frac{1}{2}$ (*d*) $\frac{1}{6} \times \frac{1}{2}$
III. (*a*) $\frac{1}{2} \times \frac{2}{3}$ (*b*) $\frac{1}{3} \times \frac{3}{4}$ (*c*) $\frac{1}{2} \times \frac{4}{5}$ (*d*) $\frac{1}{3} \times \frac{6}{7}$
IV. (*a*) $\frac{2}{3} \times \frac{1}{2}$ (*b*) $\frac{3}{4} \times \frac{1}{3}$ (*c*) $\frac{4}{5} \times \frac{1}{2}$ (*d*) $\frac{6}{7} \times \frac{1}{3}$
V. (*a*) $\frac{1}{2} \times \frac{3}{4}$ (*b*) $\frac{1}{3} \times \frac{2}{3}$ (*c*) $\frac{1}{4} \times \frac{3}{4}$ (*d*) $\frac{1}{2} \times \frac{3}{5}$
VI. (*a*) $\frac{3}{4} \times \frac{1}{2}$ (*b*) $\frac{2}{3} \times \frac{1}{3}$ (*c*) $\frac{3}{4} \times \frac{1}{4}$ (*d*) $\frac{3}{5} \times \frac{1}{2}$

It is expected, of course, that the primary drills in which only one factor is a fraction will be continued and extended to include larger numbers; covering such combinations as:

(*a*) 36, 52, 76, 96, 34, 54, 78, 92, 32, etc., $\times \frac{1}{2}$
(*b*) 51, 72, 84, 48, 81, 78, 42, 87, 54, etc., $\times \frac{1}{3}$
(*c*) 96, 69, 36, 66, 93, 63, 39, 45, 33, etc., $\times \frac{2}{3}$

and that oral and written review problems will include a mixed number either as a multiplier or a multiplicand.

Occasional sight exercises might be given as an extension of the types already presented, but without special analysis. These merely prefix an integer to one of the fractions, the integer being a multiple of the denominator of the other fraction.

(*a*) $\frac{1}{2}$ of $10\frac{1}{2}$ (*b*) $\frac{1}{2}$ of $20\frac{1}{4}$ (*c*) $\frac{1}{2}$ of $18\frac{1}{3}$
(*d*) $\frac{1}{2}$ of $12\frac{2}{3}$ (*e*) $\frac{1}{2}$ of $24\frac{4}{5}$ (*f*) $\frac{1}{2}$ of $16\frac{8}{9}$
(*g*) $\frac{1}{2}$ of $14\frac{3}{4}$ (*h*) $\frac{1}{2}$ of $22\frac{3}{5}$ (*i*) $\frac{1}{2}$ of $26\frac{3}{7}$
(*j*) $\frac{1}{3}$ of $12\frac{1}{2}$ (*k*) $\frac{1}{3}$ of $21\frac{1}{3}$ (*l*) $\frac{1}{3}$ of $30\frac{1}{4}$
(*m*) $\frac{1}{3}$ of $15\frac{3}{4}$ (*n*) $\frac{1}{3}$ of $24\frac{6}{7}$ (*o*) $\frac{1}{3}$ of $33\frac{9}{10}$
(*p*) $\frac{1}{3}$ of $18\frac{2}{3}$ (*q*) $\frac{1}{3}$ of $27\frac{4}{5}$ (*r*) $\frac{1}{3}$ of $36\frac{5}{6}$
(*s*) $\frac{1}{4}$ of $24\frac{1}{2}$ (*t*) $\frac{1}{4}$ of $32\frac{4}{5}$ (*u*) $\frac{1}{4}$ of $40\frac{3}{4}$

These should not be analyzed in any way. The children can give the correct answers and they should not be unnecessarily helped. A few blackboard exercises of somewhat similar character interest pupils at this stage; such as:

$$\begin{array}{r} (a)\ 10\frac{1}{2} \\ \times\ 2\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} (b)\ 20\frac{1}{4} \\ \times\ 4\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} (c)\ 18\frac{1}{3} \\ \times\ 3\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} (d)\ 12\frac{2}{3} \\ \times\ 3\frac{1}{3} \\ \hline \end{array}$$

$$\begin{array}{r} (e)\ 24\frac{4}{5} \\ \times\ 5\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} (f)\ 16\frac{8}{9} \\ \times\ 9\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} (g)\ 14\frac{3}{4} \\ \times\ 4\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} (h)\ 22\frac{3}{5} \\ \times\ 10\frac{1}{3} \\ \hline \end{array}$$

The integer in the multiplier should be a multiple of the denominator of the fraction in the multiplicand, so as to have one of the partial products a whole number. Examples of this kind are given only when the pupils have used a mixed number multiplicand in the fourth year. If the pupils have continued such primary sight exercises as:

$$\begin{array}{r} (i)\ 8\frac{4}{5} \\ \times\ 10 \\ \hline \end{array} \qquad \begin{array}{r} (j)\ 10\frac{1}{2} \\ \times\ 4 \\ \hline \end{array} \qquad \begin{array}{r} (k)\ 12\frac{2}{3} \\ \times\ 3 \\ \hline \end{array} \qquad \begin{array}{r} (l)\ 20\frac{2}{5} \\ \times\ 5 \\ \hline \end{array}$$

they will have no difficulty in writing in two lines the partial products in the other examples. These are made blackboard exercises in order that the pupils will employ only the figures of the two partial products and their sum, as follows:

$$(a)\ \begin{array}{r} 21 \\ +\ 5\frac{1}{4} \\ \hline 26\frac{1}{4} \end{array} \qquad (b)\ \begin{array}{r} 81 \\ +\ 10\frac{1}{8} \\ \hline 91\frac{1}{8} \end{array} \qquad (c)\ \begin{array}{r} 55 \\ +\ 9\frac{1}{6} \\ \hline 64\frac{1}{6} \end{array} \qquad (d)\ \begin{array}{r} 38 \\ +\ 4\frac{2}{9} \\ \hline 42\frac{2}{9} \end{array}$$

These interjected exercises and reviews are important in keeping up the pupils' knowledge of number, which is diminished when they give undivided attention too long to the details of a process, as is shown by the absurd answers they give to problems involving mixed numbers when they are not drilled to avoid such blunders.

The general method of finding the product of two fractions is developed through such combinations as the following:

VII. (*a*) $\frac{2}{3} \times \frac{3}{4}$ (*b*) $\frac{3}{4} \times \frac{8}{9}$ (*c*) $\frac{2}{3} \times \frac{9}{10}$

The average pupil will have no particular difficulty in analyzing (*a*) to the effect that 1 third of $\frac{3}{4}$ being $\frac{1}{4}$, 2 thirds of $\frac{3}{4}$ will be twice $\frac{1}{4}$, or $\frac{1}{2}$. When several examples of this type have been worked, the teacher should call the attention of the class to the use of cancellation to shorten exercises involving both multiplication and division, and ask some pupil to work one of the foregoing examples by first rejecting any factor common to a numerator and a denominator. The result obtained in this way, being the same as that obtained in the oral solution, satisfies the children as to the correctness of the method.

$$\begin{array}{l} 6\frac{2}{3} \times 1\frac{1}{2} \\ \underline{\ \ 3\frac{1}{3}} \\ 10\ \textit{Ans.} \end{array}$$

Without spending much time on products of two fractions, examples in which at least one of the factors is a mixed number should be presented. A pupil may be sent to the blackboard to find $1\frac{1}{2}$ times $6\frac{2}{3}$. He may indicate the work as is here shown, by writing under $6\frac{2}{3}$ its one-half and combining the two. The teacher then asks him to express both mixed numbers as improper fractions

$$\frac{\overset{10}{\cancel{20}}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}} = 10\ \textit{Ans.}$$

and to find the product of the latter. If the pupil does not cancel, he is told to do so.

A second pupil may be asked to multiply $6\frac{2}{3}$ by $2\frac{1}{2}$.

After writing the two mixed numbers with the sign of multiplication between them, he may be told to place under each its equivalent in the form of an improper fraction. He then cancels and obtains the result. He is now asked to state the method, which should take some such form as this: "The product of two mixed numbers is obtained by changing them to improper fractions, canceling common factors, writing the product of the new numerators over the product of the new denominators, and reducing the resulting improper fraction to a mixed number."

$$6\tfrac{2}{3} \times 2\tfrac{1}{2} =$$
$$\frac{\overset{10}{\cancel{20}}}{3} \times \frac{5}{\cancel{2}} = \frac{50}{3} = 16\tfrac{2}{3}\ \textit{Ans.}$$

Another pupil may be asked to multiply $\frac{2}{3}$ by $\frac{4}{5}$, and to give the rule. In this case he is led to say that the product of two fractions is obtained by placing the product of the numerators over the product of the denominators. It may then be developed that this last rule includes all forms, the two reductions and the cancellation in the preceding example being incidental.

$$\tfrac{2}{3} \times \tfrac{4}{5} = \tfrac{8}{15}\ \textit{Ans.}$$

A third pupil may multiply 84, say, by $3\frac{1}{4}$, using this method. The placing of the denominator, 1, under 84 is suggested. Advantage should be taken of an example of this kind to require the pupil to write at once the product of 21 by 13. The accuracy of the result may be tested by working the example in the usual way.

$$84 \times 3\tfrac{1}{4} =$$
$$\frac{\overset{21}{\cancel{84}}}{1} \times \frac{13}{\cancel{4}} = 273\ \textit{Ans.}$$

$$\begin{array}{r} 84 \\ \underline{3\tfrac{1}{4}} \\ 21 \\ \underline{252} \\ 273\ \textit{Ans.} \end{array}$$

When the pupils have had some practice in finding the products of two fractions, three or more factors should be employed in some exercises. A problem of this kind may be given: "Find the cost of the cloth required for 8 suits when each takes $6\frac{1}{2}$ yards, and the cloth is worth $\$1\frac{1}{4}$ per yard." The operations are thus indicated: $\$1\frac{1}{4} \times 6\frac{1}{2} \times 8$, and the result is obtained. Exercises containing larger numbers may then be used; such as $84 \times 3\frac{1}{4} \times 1\frac{1}{3}$. The answers should not always be whole numbers, nor should they necessarily be small fractions. Some of the sight examples of the lower grades should be extended to cover these types:

$$4 \times 63\tfrac{1}{2} \times 25$$
$$3 \times 47\tfrac{1}{4} \times 33\tfrac{1}{3}$$
$$8 \times 59\tfrac{1}{4} \times 12\tfrac{1}{2}$$

to accustom children to take 4 and 25, 3 and $33\frac{1}{3}$, 8 and $12\frac{1}{2}$, respectively, as one combination, notwithstanding the order in which the factors are written.

If the children have not already used mixed numbers as divisors, a preliminary lesson in division of fractions might take some such form as this:

(*a*) $2\overline{)60}$
(*b*) $\frac{1}{2}\overline{)60}$
(*c*) $\frac{3}{2}\overline{)60}$
(*d*) $3\overline{)60}$
(*e*) $1\frac{1}{2}\overline{)60}$

A pupil writes on the blackboard the dividend 60 five times. He is then told to write 2, $\frac{1}{2}$, $\frac{3}{2}$, 3, and $1\frac{1}{2}$ as the five divisors, and to find the five quotients. If he has trouble with the quotient of (*b*) he is asked how often $\frac{1}{2}$ goes into 1, into 2, into 10, into 20, etc. Before writing the quotient of (*c*) he may be asked how a quotient is affected when the divisor is doubled, the dividend remaining the same; how it is affected when the divisor is halved, the dividend remaining the same. The relation between the divisors (*b*) and (*c*) is asked, also how it affects the quotient. He should finally be able to state that the quotient of (*c*) is one-third of the quotient of (*b*), or $\frac{1}{3}$ of twice 60, or $\frac{2}{3}$ of 60. In the same way,

$$\begin{array}{r} 1\tfrac{1}{2})\,60 \\ \underline{\times 2 \times 2} \\ 3)\underline{120} \\ 40 \text{ \textit{Ans.}} \end{array}$$

Test $40 \times 1\frac{1}{2} = 60$

he should be led to state that the quotient of (*e*) is twice the quotient of (*d*), or twice ⅓ of 60, or ⅔ of 60. He may then be asked to rewrite (*e*), to multiply the divisor and the dividend by 2, and to find the quotient.

He should now be able to see that he obtains the quotient of 60 ÷ 1½ by multiplying 60 by 2 and dividing the result by 3. He is next asked to indicate the product of 60 by ⅔ and to note that it calls for the product of 60 by 2 and the division of the result by 3.

$$\frac{\overset{20}{\cancel{60}}}{1} \times \frac{2}{\cancel{3}} = 40 \text{ \textit{Ans.}}$$

Test $40 \times 1\frac{1}{2} = 60$

The same procedure is followed in the following exercises, which are written on the blackboard by another pupil, who writes the answer to each as it is given by some pupil named by the teacher. He begins by writing 12 seven times and writes each divisor as it is stated by the teacher. When a wrong answer is given, the pupil should not write it. If he writes an incorrect answer, pupils at their seats raise their hands to indicate that they have obtained a different one.

(*a*)	$3)\underline{12}$	(*h*)	$4)\underline{20}$	(*o*)	$5)\underline{30}$
(*b*)	$\frac{1}{3})\underline{12}$	(*i*)	$\frac{1}{4})\underline{20}$	(*p*)	$\frac{1}{5})\underline{30}$
(*c*)	$\frac{4}{3})\underline{12}$	(*j*)	$\frac{5}{4})\underline{20}$	(*q*)	$\frac{6}{5})\underline{30}$
(*d*)	$4)\underline{12}$	(*k*)	$5)\underline{20}$	(*r*)	$6)\underline{30}$
(*e*)	$1\frac{1}{3})\underline{12}$	(*l*)	$1\frac{1}{4})\underline{20}$	(*s*)	$1\frac{1}{5})\underline{30}$
(*f*)	$\frac{3}{4})\underline{12}$	(*m*)	$\frac{4}{5})\underline{20}$	(*t*)	$\frac{5}{6})\underline{30}$
(*g*)	$\frac{1}{4})\underline{12}$	(*n*)	$\frac{1}{5})\underline{20}$	(*u*)	$\frac{1}{6})\underline{30}$

(*a*) $12 \div 3 =$

$$\frac{\overset{4}{\cancel{12}}}{1} \times \frac{1}{\cancel{3}} = 4 \text{ \textit{Ans.}}$$

Test $4 \times 3 = 12$

After the results are all written, various pupils are called upon to test them, the information being first elicited that the product of a divisor and a quotient should produce the dividend. The rule is then developed that a number is divided by a fraction when it is multiplied by the fraction inverted. Some of the foregoing examples may be worked on the blackboard in this form by different pupils, each of whom writes the test after the solution. In (*a*) 3 is considered as the improper fraction $\frac{3}{1}$ for the purpose of illustrating the rule. The dividend 12 is also made $\frac{12}{1}$ throughout. The fact that the quotient is greater than 12 when the divisor is less than 1 should be elicited from the pupils.

(*b*) $12 \div \frac{1}{3} =$
$\frac{12}{1} \times \frac{3}{1} = 36$ *Ans.*
Test $36 \times \frac{1}{3} = 12$

(*c*) $12 \div \frac{4}{3} =$
$\frac{\overset{3}{\cancel{12}}}{1} \times \frac{3}{\cancel{4}} = 9$ *Ans.*
Test $9 \times \frac{4}{3} = 12$

(*e*) $12 \div 1\frac{1}{3} =$
$\frac{12}{1} \div \frac{4}{3} =$
$\frac{\overset{3}{\cancel{12}}}{1} \times \frac{3}{\cancel{4}} = 9$ *Ans.*
Test $9 \times 1\frac{1}{3} = 12$

(*f*) $12 \div \frac{3}{4} =$
$\frac{\overset{4}{\cancel{12}}}{1} \times \frac{4}{\cancel{3}} = 16$ *Ans.*
Test $16 \times \frac{3}{4} = 12$

If the teacher so desires, he can make the rule still more general by stating it as follows: "A number is divided by another when it is multiplied by the reciprocal of the latter." In this case the children must be informed that by the reciprocal of a number is meant 1 divided by the number. They readily give the reciprocal of 2, 3, 4, etc., as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., also the reciprocal of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., as 2, 3, 4, etc. The difficulty comes in getting some children to see that the quotient of 1 by $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{3}{5}$, etc., is $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{2}$, $\frac{5}{3}$, etc. Some understand it when presented in this way: 3 thirds ÷ 2 thirds = 3 ÷ 2 = $\frac{3}{2}$; 4 fourths ÷ 3 fourths = 4 ÷ 3 = $\frac{4}{3}$; etc. A problem like the following, worked on the blackboard by a pupil, sometimes helps him to understand that the quotient of $1 \div \frac{3}{4}$ is $\frac{4}{3}$:

"At 75 cents per yard, how much silk can be bought for a dollar?" By having him indicate the operation as an

$$75¢\underline{)100}¢$$
$$1\tfrac{25}{75} = 1\tfrac{1}{3}\ \text{(yd.)}$$

example in short division, he can see that the result is $1\frac{1}{3}$ yards, which he can change into $\frac{4}{3}$ yards. A thoughtless pupil in an upper grade, when asked how much silk at $\$\frac{3}{4}$ per yard can be bought for a dollar, will frequently give the answer as a yard and a quarter. When he is told that his result is correct if he means a yard of silk and a quarter of a dollar, he will change his answer to $1\frac{1}{3}$ yards, especially if he is asked how much 75-cent silk can be bought for 25 cents.

$$\$\tfrac{3}{4}\underline{)\$1}$$
$$\tfrac{4}{3}$$

After a pupil has learned that the quotient of 1 divided by $\frac{3}{4}$ is $\frac{4}{3}$, he can generally be made to understand that the quotient of 3 times 1 by $\frac{3}{4}$ is 3 times $\frac{4}{3}$, that the quotient of 12 times 1 by $\frac{3}{4}$ is 12 times $\frac{4}{3}$, etc.

The teacher must not, however, be discouraged at the inability of some of his pupils to master the principles underlying division of fractions. He must remain content if they get the correct answers to the exercises by merely following the rule blindly. Some will obtain the light in an upper grade, others will never quite understand the method. By requiring pupils to test the accuracy of each result, the teacher will give them all that they are able to assimilate at this time.

Special attention should be given to sight drill exercises, which may include the following types:

I. (*a*) $\frac{1}{2}\underline{)1}$ (*b*) $\frac{1}{2}\underline{)2}$ (*c*) $\frac{1}{3}\underline{)3}$ (*d*) $\frac{1}{4}\underline{)2}$ (*e*) $\frac{1}{5}\underline{)3}$
(*f*) $\frac{2}{3}\underline{)1}$ (*g*) $\frac{2}{3}\underline{)2}$ (*h*) $\frac{2}{3}\underline{)3}$ (*i*) $\frac{3}{4}\underline{)1}$ (*j*) $\frac{3}{4}\underline{)2}$
(*k*) $\frac{5}{6}\underline{)1}$ (*l*) $\frac{5}{6}\underline{)2}$ (*m*) $\frac{2}{5}\underline{)3}$ (*n*) $\frac{2}{5}\underline{)2}$ (*o*) $\frac{2}{5}\underline{)4}$

The foregoing combinations should also be presented in this form:

(*a*) $1 \div \frac{1}{2}$ (*b*) $2 \div \frac{1}{2}$ (*c*) $3 \div \frac{1}{3}$ (*d*) $2 \div \frac{1}{4}$

II. (*a*) $2\underline{)\frac{1}{2}}$ (*b*) $2\underline{)\frac{1}{3}}$ (*c*) $3\underline{)\frac{1}{2}}$ (*d*) $3\underline{)\frac{1}{3}}$ (*e*) $2\underline{)\frac{2}{3}}$

(*f*) $2\underline{)\frac{3}{4}}$ (*g*) $3\underline{)\frac{2}{3}}$ (*h*) $4\underline{)\frac{1}{2}}$ (*i*) $4\underline{)\frac{2}{5}}$ (*j*) $4\underline{)\frac{3}{5}}$

(*k*) $\frac{5}{6} \div 2$ (*l*) $\frac{4}{7} \div 3$ (*m*) $\frac{4}{7} \div 4$ (*n*) $\frac{5}{9} \div 2$

III. (*a*) $\frac{1}{2} \div \frac{1}{4}$ (*b*) $\frac{1}{2} \div \frac{1}{6}$ (*c*) $\frac{1}{3} \div \frac{1}{6}$ (*d*) $\frac{1}{2} \div \frac{1}{8}$
(*e*) $\frac{1}{4} \div \frac{1}{2}$ (*f*) $\frac{1}{6} \div \frac{1}{2}$ (*g*) $\frac{1}{6} \div \frac{1}{3}$ (*h*) $\frac{1}{8} \div \frac{1}{2}$
(*i*) $\frac{1}{2} \div \frac{1}{3}$ (*j*) $\frac{1}{3} \div \frac{1}{2}$ (*k*) $\frac{1}{2} \div \frac{1}{5}$ (*l*) $\frac{1}{5} \div \frac{1}{2}$

IV. (*a*) $1\frac{1}{2} \div \frac{1}{2}$ (*b*) $1\frac{1}{2} \div \frac{1}{3}$ (*c*) $1\frac{1}{2} \div \frac{1}{4}$ (*d*) $1\frac{1}{2} \div \frac{1}{5}$
(*e*) $1 \div \frac{2}{3}$ (*f*) $1\frac{1}{3} \div \frac{2}{3}$ (*g*) $1\frac{2}{3} \div \frac{2}{3}$ (*h*) $2 \div \frac{2}{3}$
(*i*) $1 \div \frac{3}{4}$ (*j*) $1\frac{1}{4} \div \frac{3}{4}$ (*k*) $1\frac{3}{4} \div \frac{3}{4}$ (*l*) $2 \div \frac{3}{4}$

Children should know that in dividing fractions having the same denominator, the latter may be rejected; that is, 6 sevenths ÷ 3 sevenths = 6 ÷ 3.

V. (*a*) $\frac{1}{2} \div 1\frac{1}{2}$ (*b*) $\frac{1}{4} \div 1\frac{1}{4}$ (*c*) $\frac{1}{3} \div 1\frac{1}{3}$ (*d*) $\frac{1}{5} \div 1\frac{1}{5}$
(*e*) $\frac{2}{3} \div 1\frac{1}{3}$ (*f*) $\frac{2}{3} \div 1\frac{2}{3}$ (*g*) $\frac{2}{3} \div 2\frac{1}{3}$ (*h*) $\frac{2}{3} \div 2\frac{2}{3}$
(*i*) $\frac{3}{4} \div 1\frac{1}{4}$ (*j*) $\frac{3}{4} \div 1\frac{3}{4}$ (*k*) $\frac{3}{4} \div 2\frac{1}{4}$ (*l*) $\frac{3}{4} \div 2\frac{3}{4}$

VI. (*a*) $\frac{2}{3} \div \frac{3}{4}$ (*b*) $\frac{3}{4} \div \frac{2}{3}$ (*c*) $\frac{2}{3} \div \frac{2}{5}$ (*d*) $\frac{2}{5} \div \frac{2}{3}$
(*e*) $\frac{2}{5} \div \frac{3}{4}$ (*f*) $\frac{3}{4} \div \frac{2}{5}$ (*g*) $\frac{3}{5} \div \frac{3}{4}$ (*h*) $\frac{3}{4} \div \frac{3}{5}$
(*i*) $\frac{2}{3} \div \frac{5}{6}$ (*j*) $\frac{5}{6} \div \frac{2}{3}$ (*k*) $\frac{3}{4} \div \frac{5}{6}$ (*l*) $\frac{5}{6} \div \frac{3}{4}$

The fact that it is not always necessary to reduce a mixed number dividend to an improper fraction is brought out by the following drills, which should frequently be employed:

VII. (*a*) $2\underline{)12}\frac{2}{3}$ (*b*) $3\underline{)21}\frac{3}{5}$ (*c*) $4\underline{)28}\frac{4}{7}$ (*d*) $5\underline{)30}\frac{5}{6}$

In each of these the integer in the dividend and the numerator of the fraction are multiples of the divisor. In each of the next eight, the integer in the dividend is a multiple of the divisor, but the numerator of the fraction is not.

(*e*) $2\underline{)12}\frac{1}{2}$ (*f*) $3\underline{)21}\frac{1}{2}$ (*g*) $4\underline{)28}\frac{1}{3}$ (*h*) $5\underline{)30}\frac{1}{3}$

(*i*) $2\underline{)12}\frac{3}{4}$ (*j*) $3\underline{)21}\frac{2}{3}$ (*k*) $4\underline{)28}\frac{3}{4}$ (*l*) $5\underline{)30}\frac{2}{3}$

After the pupils have divided the integer, they obtain the quotient of the fraction by multiplying its denominator by the divisor, the answers being:

(*e*) $6\frac{1}{4}$ (*f*) $7\frac{1}{6}$ (*g*) $7\frac{1}{12}$ (*h*) $6\frac{1}{15}$
(*i*) $6\frac{3}{8}$ (*j*) $7\frac{2}{9}$ (*k*) $7\frac{3}{16}$ (*l*) $6\frac{2}{15}$

In the next set, the remainder (combined with the fraction) makes an improper fraction whose numerator is a multiple of the divisor:

(*m*) $2\underline{)17}\frac{1}{3}$ (*n*) $3\underline{)16}\frac{1}{2}$ (*o*) $4\underline{)27}\frac{1}{5}$ (*p*) $5\underline{)33}\frac{3}{4}$

The quotient in (*m*) is 8 with a remainder of $1\frac{1}{3}$, or $\frac{4}{3}$; dividing $\frac{4}{3}$ by 2 gives $\frac{2}{3}$; *Ans.* $8\frac{2}{3}$. In (*n*), quotient is 5; remainder $1\frac{1}{2}$, or $\frac{3}{2}$; $\frac{3}{2} \div 3 = \frac{1}{2}$; *Ans.* $5\frac{1}{2}$. In (*o*) quotient is 6; remainder $3\frac{1}{5}$, or $\frac{16}{5}$; $\frac{16}{5} \div 4 = \frac{4}{5}$; *Ans.* $6\frac{4}{5}$. In (*p*) quotient is 6; remainder $3\frac{3}{4}$, or $\frac{15}{4}$; $\frac{15}{4} \div 5 = \frac{3}{4}$; *Ans.* $6\frac{3}{4}$.

In the next set the remainder (combined with the fraction) makes an improper fraction whose numerator is not a multiple of the divisor:

(*q*) $2\underline{)17}\frac{3}{4}$ (*r*) $3\underline{)16}\frac{2}{3}$ (*s*) $4\underline{)27}\frac{2}{5}$ (*t*) $5\underline{)33}\frac{1}{4}$

The quotient in (*q*) is 8 with a remainder of $1\frac{3}{4}$, or $\frac{7}{4}$; $\frac{1}{2}$ of $\frac{7}{4}$ is $\frac{7}{8}$; *Ans.* $8\frac{7}{8}$. In (*r*) quotient is 5; remainder $1\frac{2}{3}$, or $\frac{5}{3}$; $\frac{1}{3}$ of $\frac{5}{3}$ is $\frac{5}{9}$; *Ans.* $5\frac{5}{9}$. In (*s*) quotient is 6; remainder $3\frac{2}{5}$, or $\frac{17}{5}$; $\frac{1}{4}$ of $\frac{17}{5}$ is $\frac{17}{20}$; *Ans.* $6\frac{17}{20}$. In (*t*) quotient is 6; remainder $3\frac{1}{4}$, or $\frac{13}{4}$; $\frac{1}{5}$ of $\frac{13}{4}$ is $\frac{13}{20}$; *Ans.* $6\frac{13}{20}$.

Longer exercises of this kind should be occasionally placed on the blackboard and only the answers written, the pupils writing one figure at a time. The following are merely extensions of some of the foregoing. The teacher should make many more, using divisors up to 13.

(*a*) $2)\underline{8512}\,\frac{2}{3}$ (*b*) $3)\underline{5721}\,\frac{3}{5}$ (*c*) $4)\underline{9328}\,\frac{4}{7}$

(*d*) $3)\underline{8421}\,\frac{2}{3}$ (*e*) $4)\underline{6128}\,\frac{3}{4}$ (*f*) $5)\underline{8230}\,\frac{2}{3}$

(*g*) $2)\underline{5917}\,\frac{1}{3}$ (*h*) $3)\underline{8716}\,\frac{1}{2}$ (*i*) $4)\underline{9227}\,\frac{1}{5}$

(*j*) $3)\underline{4816}\,\frac{2}{3}$ (*k*) $4)\underline{7627}\,\frac{2}{5}$ (*l*) $5)\underline{6033}\,\frac{1}{4}$

When the course of study calls for division by factors, pupils readily work such examples as the following as blackboard exercises, writing in this case the two successive quotients.

(*a*) $35)\underline{8512}$ (*b*) $36)\underline{5721}$ (*c*) $42)\underline{9328}$

(*d*) $48)\underline{8421}$ (*e*) $49)\underline{6128}$ (*f*) $54)\underline{8230}$

(*g*) $56)\underline{5917}$ (*h*) $63)\underline{8716}$ (*i*) $64)\underline{9227}$

(*j*) $66)\underline{4816}$ (*k*) $72)\underline{7627}$ (*l*) $81)\underline{6033}$

$9)\underline{6033}$
$\quad 9)\underline{670}\,\frac{3}{9}$
Ans. $74\ (\frac{39}{81})\ \frac{13}{27}$
Quotient, 74;
Remainder, 39.

Each remainder is written as the numerator of a fraction, which is not reduced, in order that the final denominator may be the same as the original divisor, the numerator of the fraction then corresponding with the remainder. If the answer is required in the form of a fraction, the reduction can be performed at the end.

When the pupils have learned in the grade below to change a mixed number divisor to a whole number, they should be permitted to continue the practice; but it is unnecessary and inadvisable to introduce this method as an alternative one in written work. It employs the same figures in another way and does not take advantage of possible cancellations. When this form is continued, it may be necessary to have the pupils understand the remainders obtained when the division is inexact. In finding the number of yards that can be bought for $1 when the price per yard is $18\frac{3}{4}$ cents, the pupil may give the result as $5\frac{1}{3}$ yards. When, however, the number of calves at \$$18\frac{3}{4}$ each that can be purchased for $100 is required, the answer needs a different form. To understand the meaning of the remainder, the pupil must remember that his divisor in the accompanying example is 75 quarter dollars, and his dividend is 400 quarter dollars, the quotient being 5 and the remainder 25 quarter dollars. The quotient indicates the *number* of calves, and this word is supplied in writing the result, the 25 quarter dollars being changed to \$$6\frac{1}{4}$. The answer, $5\frac{1}{3}$ calves, can be changed to the foregoing by changing $\frac{1}{3}$ calf to $\frac{1}{3}$ of \$$18\frac{3}{4}$.

\$$18\frac{3}{4}$)$100
5
75 quarter dollars)400 quarter dollars
375 " "
25 quarter dollars
\$ $6\frac{1}{4}$
Ans. 5 calves, \$$6\frac{1}{4}$ rem.

When the answer is to be given as a mixed number, the denomination is omitted both from the divisor and the dividend, the fraction $\frac{25}{75}$ being the same as $\frac{\text{25 quarter dollars}}{\text{75 quarter dollars}}$, both being equal to $\frac{1}{3}$.

Problems with fractional divisors calling for remainders should be postponed as long as possible. They sometimes occur in the sixth year, in the reduction, for instance, of say 119 yards to rods. By testing answers of this kind, the pupil should detect any mistakes he may have made.

11 half yards)238 half yards
21 (rods) 7 half yards
Ans. 21 rods $3\frac{1}{2}$ yards
or, 21 rd. 3 yd. 1 ft. 6 in.

The problems worked at this time should consist largely of miscellaneous examples in multiplication or division that involve only one step. They should contain the various conditions with which the children are familiar, but the wording should not expressly state the operation.

The following are some typical problems that may be written on the blackboard if the text-books do not contain similar ones:

(*a*) "What fraction of $2\frac{3}{5}$ is $4\frac{5}{8}$?"

(*b*) "Find $\frac{3}{4}$ of $27\frac{2}{3}$."

(*c*) "Give the cost of a yard of silk when $\frac{3}{4}$ yard costs $43\frac{3}{4}$ cents."

(*d*) "$14\frac{2}{3}$ is $\frac{2}{5}$ of what number?"

(*e*) "$\frac{5}{12}$ is what fraction of $\frac{4}{15}$?"

(*f*) "$\frac{4}{15}$ is what fraction of $\frac{5}{12}$?"

(*g*) "How many square yards are there in a strip of carpet 56 yards long, $\frac{5}{9}$ yard wide?"

(*h*) "If it requires $\frac{7}{8}$ yard of material to make an apron, how many aprons can be made from 196 yards?"

(*i*) "When velvet is $3.85 per yard, what part of a yard can be bought for $1.54?"

(*j*) "A roll of carpet contains $28\frac{1}{8}$ square yards. If it is $\frac{3}{4}$ yard wide, how long is it?"

(*k*) "How many acres of land has a farmer when there are 225 acres in $\frac{15}{17}$ of his farm?"

(*l*) "How many baseballs at $\$\frac{3}{16}$ each can be bought for $72?"

(*m*) "If two men can do $\frac{43}{120}$ of a piece of work in a week, how many weeks would they require to do the whole work?"

These should be preceded by simple examples of the same types, in answering each of which the pupil should state the operation employed in obtaining the result, using the numbers given.

The average child does not realize that the following oral problem is one in division:

"Find the cost of a yard of ribbon when $\frac{1}{2}$ yard costs 8 cents."

Although in the lower grades he may be permitted to obtain the answer to a written problem in his own way, he should be required in the grammar grades to indicate the operation before commencing to do the work.

When the foregoing type is presented with the following numbers, he recognizes it as one in division in each case:

Find the cost of a yard of ribbon when (*n*) 2 yards cost 16 cents, (*o*) $1\frac{1}{2}$ yards cost 12 cents, (*p*) 3 yards cost 24 cents, (*q*) $2\frac{1}{2}$ yards cost 20 cents, etc.

His analysis of the problem at the top of the page should also be to the effect that the number of cents in the cost of 1 yard will be the quotient of 8 divided by $\frac{1}{2}$. He may supplement this by saying that this quotient is obtained by multiplying 8 by 2, if the teacher so desires.

When the cost of $\frac{3}{4}$ yard of silk is given as 36 cents, a pupil asked to find the cost of a yard should at this recitation, at least, give some such analysis as this: "The cost of a yard will be the quotient of 36 cents by $\frac{3}{4}$." He may add, if he wishes: "This is the same as the product of 36 cents by $\frac{4}{3}$," etc.

While some teachers prefer to have their pupils first obtain the cost of $\frac{1}{4}$ yard by taking $\frac{1}{3}$ of 36 cents, or 12 cents, and to find the cost of a yard by multiplying 12 cents by 4, there are many others who realize that a great number of pupils able to state the analysis in this way do not grasp its meaning.

This form suits a problem manufactured for the purpose, but it does not lend itself to such as contain fractions or have fractional answers. As the given example can be directly solved by a single division, the analysis of the oral problem should aid similar written ones by naming the process.

Before the pupils solve the written problems on page 188, there should be a preliminary oral recitation in which they are discussed. In (*a*) the teacher advises the substitution of whole numbers, and a pupil reads the problem: "What fraction of 2 is 4?" As he is accustomed to think the term "fraction" implies a proper one, he is inclined to give the answer as $\frac{1}{2}$; but the correct result is finally brought out that 4 is twice 2, and that the answer to the original problem may be written as an improper fraction, obtained by dividing $4\frac{5}{8}$ by $2\frac{3}{5}$, or by simplifying the corresponding complex fraction. In (*c*) the problem may be read without the denominator of the first fraction, in order that the pupil may ascertain the operation: "Give the cost of a yard when 3 yards cost $43\frac{3}{4}$ cents." When this shows him that division is called for, he reads it again as it is given, and states that the price is $43\frac{3}{4}$ cents divided by $\frac{3}{4}$. By first reading (*d*) as "$14\frac{2}{3}$ is twice what number?" a pupil recognizes the problem as one in division, and he states that "$14\frac{2}{3}$ is divided by $\frac{2}{5}$." Since in (*e*), both numbers are fractions, the careless pupil is prone to make the larger one the divisor, even when he reads it, "5 is what fraction of 4?" It may be advisable at this point to develop the fact that "of" indicates multiplication, and that the original example may be read "$\frac{5}{12}$ is how many times $\frac{4}{15}$?" If the term "fraction" in examples of this kind always indicates an answer less than 1, the pupils need only to make the larger number the divisor, and they are likely to do the same in the seventh year in percentage examples of a similar type. By making his pupils understand that a fraction in the answer may be an improper one, and by using the word "times" occasionally even when the answer is a proper fraction, the teacher may overcome the tendency to carelessness on the part of some pupils.

When the pupils work the foregoing problems, they begin each by indicating the operation; as follows:

(*a*) $4\frac{5}{8} \div 2\frac{3}{5}$ (*b*) $\frac{3}{4} \times 27\frac{2}{3}$ (*c*) $43\frac{3}{4}$ ¢ $\div \frac{3}{4}$
(*d*) $14\frac{2}{3} \div \frac{2}{5}$ (*e*) $\frac{5}{12} \div \frac{4}{15}$ (*g*) $(56 \times \frac{5}{9})$ sq. yd.
(*h*) $(196 \div \frac{7}{8})\ a.$ (*i*) $(154 \div 385)$yd. (*j*) $(28\frac{1}{8} \div \frac{3}{4})$ yd.
(*k*) $225\text{A} \div \frac{15}{17}$ (*l*) $(72 \div \frac{3}{16})b$ (*m*) $(1 \div \frac{43}{120})$ wk.

Each answer should be checked, (*a*) by multiplying $4\frac{5}{8}$ by it, which should give $2\frac{3}{5}$. All are tested by multiplication, except (*b*) and (*g*). These may be proved by working them again by division. The employment of books containing answers, unfortunately makes tests unnecessary, and also indirectly shows the pupil when division rather than multiplication is required.

In connection with the determination of the process, the discussion of the problem may be accompanied by an approximation of the result, but the latter should not be allowed to obscure the main object. The use of small numbers or of whole numbers tends to simplify some problems so as to bring more clearly to the mind of the pupil the fact that he should multiply or divide, according to the conditions. These substituted numbers need not necessarily have any particular relation to those of the problem. After the operation has been determined, the approximation may follow.

The analysis suggested for the oral problems intended to develop the process to be employed, should not be required in the case of problems given at other times. In the latter the result is of chief importance, and pupils should be given a choice in the selection of the method.

In the following, for instance, the pupil may employ the ratio method:

"When $\frac{2}{5}$ of a number is 16, what is $\frac{4}{5}$ of the number?"

His analysis in this case may be: "Since 2 fifths of the number is 16, 4 fifths of it is twice 16, or 32," without being required to bring out the fact that 4 fifths are twice 2 fifths. In a written example, giving, for instance, $\frac{2}{15}$ of a number as $64\frac{3}{4}$ and requiring $\frac{8}{15}$ of it, the indicated operations should, however, include all of the numbers given, and take one of these forms:

$$64\tfrac{3}{4} \times (\tfrac{8}{15} \div \tfrac{2}{15}),\quad (64\tfrac{3}{4} \div \tfrac{2}{15}) \times \tfrac{8}{15},\quad \text{etc.}$$

Although it is unwise to require the pupils to employ unnecessary words or figures in their written solutions, the practice of indicating the required processes does not fall under this ban. Having shown, as given above, the parts played by $\frac{2}{15}$, $64\frac{3}{4}$, and $\frac{8}{15}$, they should be permitted to write the result at once as 259 *Ans.*, which they obtain by multiplying $64\frac{3}{4}$ by 4 without again writing either of these numbers.

A teacher who desires his pupils to find the whole when $\frac{2}{3}$ is given as 24, for instance, by first dividing 24 by 2 to obtain $\frac{1}{3}$, etc., should include among his oral exercises problems involving the following:

When $\frac{1}{8} = 12$, find (*a*) $\frac{3}{8}$, (*b*) $\frac{5}{8}$, (*c*) $\frac{7}{8}$, (*d*) $\frac{1}{2}$, (*e*) $\frac{1}{4}$, (*f*) $\frac{3}{4}$; and the corresponding reversed types: When (*g*) $\frac{3}{8} = 36$, (*h*) $\frac{5}{8} = 60$, (*i*) $\frac{7}{8} = 84$, (*j*) $\frac{1}{2} = 48$, (*k*) $\frac{1}{4} = 24$, (*l*) $\frac{3}{4} = 72$, find $\frac{1}{8}$.

Each of these requires but one step for its solution by the ratio method, as follows:

(*a*) 12×3, (*b*) 12×5, (*c*) 12×7, (*d*) 12×4, (*e*) 12×2, (*f*) 12×6

(*g*) $36 \div 3$, (*h*) $60 \div 5$, (*i*) $84 \div 7$, (*j*) $48 \div 4$, (*k*) $24 \div 2$, (*l*) $72 \div 6$

The teacher may, however, occasionally require a pupil to state how he ascertains the multiplier or the divisor he employs; in (*d*), for instance, to explain why he multiplies 12 by 4, the reply that $\frac{1}{2}$ is 4 times $\frac{1}{8}$ being desired.

Other exercises are then used which involve the comparison of $\frac{2}{9}$ with $\frac{4}{9}$, $\frac{2}{3}$, and $\frac{8}{9}$; $\frac{2}{5}$ with $\frac{4}{5}$, etc.; in which one fraction is an integral number of times the other.

The first oral problem that asks the finding of the whole when the value of $\frac{2}{3}$ is given should be given in two parts by the teacher who desires the two-step analysis, thus:

"When 2 thirds of a yard of ribbon cost 24 cents, (*a*) what is the cost of 1 third of a yard? (*b*) What is the cost of a yard?"

This seems a long-drawn-out procedure, but experience shows that a pupil frequently says $\frac{1}{2}$ is $\frac{1}{3}$ of 24, without comprehending the fact that he is really stating an equation: $\frac{1}{2}x = \frac{24}{3}$, of which $\frac{1}{2}$ of something is the first member and $\frac{1}{3}$ of 24 is the second. If this pupil thinks at all, he may wonder why he says $\frac{1}{2}$ is $\frac{1}{3}$, but he does not attempt to understand it further than to appreciate the fact that many problems seem to call for this sort of thing. Some of the difficulty can be prevented by omitting the use of one-half, analyzing in this way:

"When 2 thirds of a yard cost 24 cents, 1 third of a yard will cost 24 cents divided by 2, or 12 cents; etc."

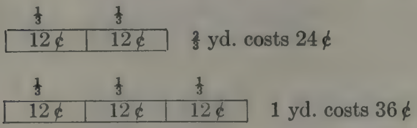
The same problem should be presented in another way by one of the pupils, who is asked to express it in feet, and to find the answer:

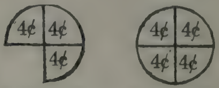
"When 2 feet of ribbon cost 24 cents, (*a*) what is the cost of 1 foot? (*b*) What is the cost of 3 feet?"

Another may be asked to state the problem in inches, as follows:

"When 24 inches of ribbon cost 24 cents, (*a*) what is the cost of 1 inch, (of 12 inches)? (*b*) What is the cost of 36 inches?"

The average fifth-grade teacher does not always appreciate the tribulations of the one in the seventh grade. The former frequently assumes that his pupils understand this type of problem because their answers are generally correct; but the work a couple of years later with the same fractions expressed as per cents shows that the contrary is the case. The employment of an occasional diagram serves to illustrate the process in a way that appeals to some children. The cost of a pie when 3 quarters cost 12 cents is shown by drawing 3 quarters of a circle and marking in each quarter its cost. A whole pie is also drawn, divided into 4 quarters, the price of each marked in it, and the cost of the pie written underneath.





¾ pie, 12¢ 1 pie, 16¢
¼ pie, 4¢

Oral problems involving the comparison of two fractions are frequently simplified by reducing both to equivalent fractions having a common denominator, and rejecting the latter. If, for instance, it is required to determine what fraction of ⅘ is ⅔, the pupil changes the problem to "What fraction of 12 fifteenths is 10 fifteenths, or what fraction of 12 is 10?" to which he obtains the answer $\frac{10}{12}$, which he reduces to ⅚. In the same way, when the cost of ⅔ is given and the cost of ¾ is required, he changes the former to 8 twelfths and the latter to 9 twelfths, then rejects the twelfths and substitutes 8 and 9 for the respective fractions.

There is no need of dwelling upon compound or complex fractions further than to state that ½ of ¾ is called a compound fraction, which consists of a fraction of a fraction; and that $\frac{1\frac{1}{2}}{2}$, $\frac{2}{3\frac{3}{4}}$, and $\frac{6\frac{3}{4}}{7\frac{1}{2}}$ are called complex fractions, which have a fraction in either term or in both. The pupils already know that a compound fraction is an indicated multiplication, and they recognize from its form that a complex fraction is an indicated division. Drills in division of fractions may include some complex forms, each of which is simplified by multiplying both terms by the least common denominator of the fractions when there are two, and reducing the result to lowest terms. The sight drills may begin with those having one fractional term and then take up those having fractions in both terms, as follows:

I. (*a*) $\frac{1\frac{1}{2}}{2}$ (*b*) $\frac{2}{2\frac{1}{2}}$ (*c*) $\frac{3}{4\frac{1}{2}}$ (*d*) $\frac{1\frac{3}{4}}{2}$ (*e*) $\frac{2\frac{3}{4}}{3}$ (*f*) $\frac{3}{3\frac{1}{8}}$

II. (*a*) $\frac{1\frac{1}{4}}{2\frac{1}{2}}$ (*b*) $\frac{1\frac{1}{4}}{2\frac{2}{3}}$ (*c*) $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ (*d*) $\frac{2\frac{2}{3}}{4\frac{1}{5}}$ (*e*) $\frac{3\frac{1}{2}}{4\frac{1}{2}}$ (*f*) $\frac{4\frac{1}{2}}{6\frac{3}{4}}$

There should be a few written exercises that may be worked by multiplying both terms by the least common denominator of the fractions when it is a convenient number:

$$\frac{3\frac{3}{4}}{5\frac{5}{6}} = \frac{3\frac{3}{4} \times 12}{5\frac{5}{6} \times 12} = \frac{45}{70} = \frac{9}{14} \text{ Ans.}$$

$$\frac{2\frac{1}{10}}{5\frac{2}{15}} = \frac{2\frac{1}{10} \times 30}{5\frac{2}{15} \times 30} = \frac{63}{154} = \frac{9}{22} \text{ Ans.}$$

As a rule, time is saved by considering the reduction as an example in division of fractions; thus:

$$\frac{4\frac{7}{8}}{15\frac{1}{6}} = 4\tfrac{7}{8} \div 15\tfrac{1}{6} = \frac{\overset{3}{\cancel{39}}}{\underset{4}{\cancel{8}}} \times \frac{\overset{3}{\cancel{6}}}{\underset{7}{\cancel{91}}} = \frac{9}{28} \text{ Ans.}$$

All of the fractions in an example involving only multiplication and division should be changed to the multiplication form by inverting the divisors and changing the division sign to one of multiplication; thus:

$3\frac{3}{4} \div \frac{7}{8} \div 3\frac{1}{5} \times \frac{14}{15}$ is changed to $\frac{15}{4} \times \frac{8}{7} \times \frac{5}{16} \times \frac{14}{15}$, the mixed numbers being reduced at the same time to improper fractions. Care must be taken with examples expressed in either of the following ways:

(*a*) $3\frac{3}{4} \div (\frac{7}{8} \div 3\frac{1}{5}) \times \frac{14}{15}$. This should first be rewritten to make the divisor in the parenthesis, $3\frac{1}{5}$, a multiplier; thus: $3\frac{3}{4} \div (\frac{7}{8} \times \frac{5}{16}) \times \frac{14}{15}$, and after the removal of the parenthesis $= 3\frac{3}{4} \times \frac{8}{7} \times \frac{16}{5} \times \frac{14}{15}$.

(*b*) $\dfrac{3\frac{3}{4} \times \frac{14}{15}}{\frac{7}{8} \div 3\frac{1}{5}}$ should also be first changed to $\dfrac{\frac{15}{4} \times \frac{14}{15}}{\frac{7}{8} \times \frac{5}{16}}$, the divisor in the denominator being made a multiplier. The next step inverts the fractions below the line, giving the following: $\frac{15}{4} \times \frac{14}{15} \times \frac{8}{7} \times \frac{16}{5}$.

If the pupil has any doubt about the correct procedure, he should first perform the operation upon the fractions within the parenthesis, thus combining them into one fraction. This he has to do when the fractions within the parenthesis or those above or below the line are connected by a + or a − sign, as in the following:

Simplify: (*a*) $3\frac{3}{4} \div (\frac{7}{8} + 3\frac{1}{5}) \times \frac{14}{15}$

(*b*) $\dfrac{3\frac{3}{4} \times \frac{14}{15}}{3\frac{1}{5} - \frac{7}{8}}$

Examples of this kind should not be too complicated.

As has been said in another connection, the written problems in the text-books should be solved by the pupils at their seats after a preliminary oral discussion to bring out the method of solving each. Problems given by the teacher to supplement those in the book should be simple, and each should be read twice. At the first reading, the pupils should attend merely to the conditions; at the second, they should write the given numbers and also indicate the necessary operations by signs. The numbers may be written on a line or arranged in the form employed in doing the work.

The later problems should be limited to two steps; as, for instance:

(*a*) "How many square yards are there in two pieces of cloth, each 27 inches wide, when one contains $36\frac{5}{8}$ linear yards and the other $42\frac{3}{4}$ linear yards?"

(*b*) "Mr. Ziegler had $48\frac{4}{5}$ acres of land. How many acres had he after selling $27\frac{3}{4}$ acres and buying $15\frac{7}{10}$ acres?"

(*c*) "If it requires $36\frac{1}{2}$ square yards of carpet to cover a floor when the carpet is $\frac{3}{4}$ yard wide, how many square yards will be needed when the carpet is $\frac{8}{9}$ yard wide?"

(*d*) "How many acres of land are there in two fields if the first contains $40\frac{3}{4}$ acres and the second contains $10\frac{5}{8}$ acres more than the first?"

(*c*) "After traveling $3\frac{1}{2}$ hours at the rate of $12\frac{1}{2}$ miles per hour, how many miles has a man yet to go to travel 50 miles?"

The operations may be indicated thus:

(*a*) $[36\frac{5}{8}\ (\text{yd.}) + 42\frac{3}{4}\ (\text{yd.})] \times \frac{3}{4}\ (\text{yd.})$

(*b*) $48\frac{4}{5}\text{ A.} - 27\frac{3}{4}\text{ A.} + 15\frac{7}{10}\text{ A.}$

(*c*) $36\frac{1}{2}\text{ sq. yd.} \times \frac{3}{4} \div \frac{8}{9}$

(*d*) $40\frac{3}{4}\text{ A.} + 40\frac{3}{4}\text{ A.} + 10\frac{5}{8}\text{ A.}$

(*e*) $50\text{ mi.} - (12\frac{1}{2}\text{ mi.} \times 3\frac{1}{2})$

Some pupils may employ the following arrangement or something similar:

(*a*)
$$\begin{array}{r} 36\frac{5}{8}\text{ yd.} \\ +\ 42\frac{3}{4}\text{ yd.} \\ \hline *\ (\text{yd.}) \times \frac{3}{4}\ (\text{yd.}) \end{array}$$

(*b*)
$$\begin{array}{r} 48\frac{4}{5}\text{ A.} \\ -\ 27\frac{3}{4} \\ \hline *\text{ A.} \\ +\ 15\frac{7}{10} \\ \hline \text{A.} \end{array}$$

(*c*)
$$\frac{36\frac{1}{2}\text{ sq. yd.} \times \frac{3}{4}}{\frac{8}{9}}$$

(*d*)
$$\begin{array}{r} 40\frac{3}{4}\text{ A.} \\ 40\frac{3}{4} \\ +\ 10\frac{5}{8} \\ \hline \text{A.}\quad \textit{Ans.} \end{array}$$

(*e*)
$$\begin{array}{r} 12\frac{1}{2}\text{ mi.} \\ \times\ 3\frac{1}{2} \\ \hline \end{array}$$

Ans. $50\text{ mi.} - *\text{ mi.}$

The answer and the manner of obtaining it may be discussed as each problem is finished, or the discussion of the results may be postponed until the set is completed. The problems should, when necessary, be worked on the blackboard by pupils who have failed to obtain the correct answers, suggestions as to better methods of arrangement and forms of solution being offered by other pupils, etc.

Special attention should be given to the examples employing multiplication and division, such as the following:

"Find the value of $\frac{4}{15}$ of an acre when $\frac{5}{12}$ of an acre is worth 57\frac{1}{2}$."

In solving this on the blackboard the form employed in similar examples containing only whole numbers is favored by many teachers. Having written 57\frac{1}{2}$ above a line and $\frac{5}{12}$ A. in front of it, they require the pupils to indicate the value of 1 A. by writing $\frac{5}{12}$ below the line (as a divisor) and the value of $\frac{4}{15}$ A. by writing $\frac{4}{15}$ above the line as a multiplier. The next step is to change 57$\frac{1}{2}$ to an improper fraction, $\frac{115}{2}$, to invert $\frac{5}{12}$, placing 12 above the line and 5 below, and to place below the line the denominator of $\frac{4}{15}$. The possible cancellations are then made, and the result obtained.

$$\begin{array}{lll} \frac{5}{12} & \text{A.} & \dfrac{\$57\frac{1}{2}\times\frac{4}{15}}{\frac{5}{12}} \\ 1 & \text{``} & \\ \frac{4}{15} & \text{``} & \end{array}$$

$$\frac{\$115\times12\times4}{2\times5\times15}$$

While the pupils may be permitted to employ special methods in solving oral problems of this type, the written problems should be indicated in the manner shown above, or on a line, thus: $\$57\frac{1}{2}\div\frac{5}{12}\times\frac{4}{15}$. This may then be rewritten $\$\frac{115}{2}\times\frac{12}{5}\times\frac{4}{15}$, etc.

The oral problems employ the same processes, but they first combine two of the numbers before operating with the third. This is shown by the following problem:

"Find the cost of 1$\frac{1}{3}$ acres of land at the rate of $30 for $\frac{3}{10}$ acre."

The required operations may be indicated thus:

$$\$30 \div \tfrac{3}{10} \times 1\tfrac{1}{5}$$

Some pupils, realizing that $1\frac{1}{5}$ is four times $\frac{3}{10}$, combine the foregoing numbers in this way:

$\$30 \times (1\frac{1}{5} \div \frac{3}{10})$, substituting 4 for the quantity in the parenthesis, which makes the final step $\$30 \times 4$.

Others make this combination: $(\$30 \div \frac{3}{10}) \times 1\frac{1}{5}$, substituting \$100 for the quantity in the parenthesis, which makes the next step $\$100 \times 1\frac{1}{5}$.

While a few oral problems of this kind are useful in developing alertness on the part of the brighter children, time should not be taken from more important details for the solution of problems that are not particularly practical. Opportunity will be found in some of the oral problems in calculating interest to show how two of three factors may be combined to shorten the work.

CHAPTER IV

DECIMALS

THE regular work in decimals, which generally constitutes the chief topic of the first half of the sixth year, is preceded as a rule by exercises in three-place decimals in the fifth year, and some courses of study provide for instruction in two-place decimals in connection with the lessons in U. S. money of the fourth year.

The place value of figures in the Arabic system of notation should be taught in connection with three-place decimals in the fifth grade. A number such as $111.11 may be used as an illustration, from which the pupils give the value of each 1, beginning at the left, the first representing 1 100-dollar bill, for instance, the next 1 10-dollar bill, the next 1 1-dollar bill, the next representing 1 dime (one-tenth of a dollar, and the last 1 cent (one-hundredth of a dollar).

Numerous sight exercises in reducing common fractions to decimals and decimals to common fractions should be given. The fact that in two-place decimals the latter are reducible to lower terms, only when they are multiples of 2 or of 5, should be brought out, and that the greatest common divisor in such a case is either 2, 4, 5, or 25. The reduction of three-place decimals should be made blackboard exercises, the greatest common divisor being 2, 4, 8, 5, 25, or 125. Work of this kind is useful in developing the number sense. Under no circumstances should it be necessary to find the greatest common divisor in any other way than by inspection.

In the few longer decimals given for practice in the higher grades, the pupils may perform the reduction in successive steps, being expected, however, to use divisors as large as possible each time. The value of .3175, for instance, should be obtained by multiplying 31 by 4 and adding 3 to the product for the numerator, making the result $\frac{127}{400}$, thus using 25 as a divisor instead of dividing by 5 twice.

Drills in familiarizing pupils with the value, in common fractions, of .125, .375, .625, and .875 in the fifth year should be extended in the sixth year to include the sixteenths: .0625, .1875, .3125, etc.

The reduction of a common fraction to a decimal should be considered as the finding of the quotient of an indicated division. Thus $\frac{7}{64}$ is changed to a decimal by dividing 7 by 64. An example of this kind is not given until the sixth year, by which time pupils have learned to find decimal quotients. The reduction of a common fraction to a three-place decimal in the fifth year is largely sight work, the denominators being 8, 40, 125, 250, or 500.

```
      .109375
   64)7.
      64
       600
       576
        24
        etc.
```

In obtaining the answers to the following, the pupil may be led to see that he multiplies both terms by such a number as will make the denominator 1000; thus:

$$\frac{4}{125} = \frac{4 \times 8}{125 \times 8} = \frac{32}{1000}, \text{ or } .032$$

$$\frac{23}{250} = \frac{23 \times 4}{250 \times 4} = \frac{92}{1000}, \text{ or } .092$$

$$\frac{109}{500} = \frac{109 \times 2}{500 \times 2} = \frac{218}{1000}, \text{ or } .218$$

$$\frac{17}{40} = \frac{17 \times 25}{40 \times 25} = \frac{425}{1000}, \text{ or } .425$$

In reducing the last, his attention is called to the fact that the same result is obtained more easily by dividing 17 by 40.

$$\begin{array}{rcr} 3.5 & = & 3.500 \\ 14.08 & = & 14.080 \\ .127 & = & .127 \\ \underline{3.009} & = & \underline{3.009} \\ \textit{Ans.}\ 20.716 & = & 20.716 \end{array}$$

In an early example in adding mixed decimals, the addends may be made of the same denomination, as shown in the accompanying example, in which each decimal is written as thousandths in the second column. This serves to emphasize the fact that terminal ciphers do not change the value of the original decimals, and also that it is unnecessary to affix them in addition or subtraction examples.

Multiplication of decimals is commenced in the fifth year, or earlier, by examples in which only one of the factors is a decimal. Pupils who have multiplied a number composed of dollars and cents by an integer readily learn to write a decimal point in the product when it is reached in the multiplicand. By degrees the fact is developed that the product must consist of the same denomination as the multiplicand when the multiplier is an integer.

$$\begin{array}{r} 18.375 \\ \underline{\times 6} \\ 110.250 \\ 110.25\ \textit{Ans.} \end{array}$$

An example containing an integral multiplicand and a decimal multiplier constitutes the second type. By this time the pupils understand that the product of 184 by .45 is equal to that of .45 by 184, and that the method of pointing-off the product of the former is the same as in the latter. The insertion of the decimal point in the partial products is unnecessary. The superfluous terminal cipher is canceled, or may be omitted in the second writing of the product.

$$\begin{array}{l} 184 \\ \underline{.45} \\ 920 \\ \underline{736} \\ 82.80 \\ 82.8\ \textit{Ans.} \end{array}$$

$$\begin{array}{r} 2.34 \\ \underline{34.5} \\ 1170 \\ 936 \\ \underline{702} \\ 80.730 \\ 80.73 \text{ } \textit{Ans.} \end{array}$$

The reason underlying the rule for pointing-off in a product the sum of the decimal places in the multiplier and the multiplicand may be shown by rewriting the factors as fractions, and working the example as multiplication of fractions. In pointing-off a product, the pupil must be required to consider also the integral portion of the result. In the foregoing example, for instance, he should know that this part of the product consists of two figures. This plan of checking the probable correctness of a result should be applied to all forms of arithmetical work.

$$\frac{234}{100} \times \frac{345}{10} = \frac{80730}{1000}$$

Many sight and blackboard exercises should be given in which the pupils employ a common fraction to obtain the result instead of the decimal that constitutes one of the factors; such as:

(*a*) 124	(*b*) 248	(*c*) .375	(*d*) .0625
× .25	× .125	× 24	× 176

Modifications of the foregoing may assume the following forms:

(*a*) 124	(*b*) 248	(*c*) 37.5	(*d*) 6.25
× 2.5	× 12.5	× 24	× 176

A pupil of this grade should be able to see that the product of 124 by 2.5 is 10 times its product by .25, and that if the latter is $\frac{1}{4}$ of 124, or 31, the former is 310.

While work of this kind should not be overdone, it should by no means be entirely omitted. When the teacher of a given grade finds that it has not been previously done, he should limit himself at the start to the simpler types, increasing the difficulties very gradually.

The work of the pupils at the blackboard frequently affords an opportunity to the teacher to require a pupil to replace some circuitous method by a more direct one.

The so-called blackboard exercises should be employed to develop the habit of employing short-cuts. Examples classified in the books as written exercises may be used for this purpose, the teacher designating the example, and the pupils writing only the answer, one figure at a time. These may be supplemented by others placed upon the blackboard by the teacher.

The introduction to work of this kind in the sixth year might be made by having the product written on the blackboard by one pupil, as another at his seat announces the successive figures. A few examples worked in this way show the class the method.

$$\begin{array}{r} 12486 \times .125 \\ \hline 1560.75 \end{array}$$

An occasional blackboard exercise may require the use of a decimal instead of the given common fraction; as, for instance:

(*a*) $246 \times \frac{2}{5}$ (*b*) $123 \times 1\frac{1}{5}$ (*c*) $321 \times \frac{3}{25}$

In these the pupil mentally changes the second factor to .4, 1.2, and .12, respectively.

When one of the factors is an integer terminating in one or more ciphers, and the other contains a decimal, some teachers shift the decimal point to the right and cancel a corresponding number of ciphers:

$$\begin{array}{r} (a)\ 8000 \\ \times .245 \\ \hline \end{array} \qquad \begin{array}{r} (b)\ 37.25 \\ \times 300 \\ \hline \end{array} \qquad \begin{array}{r} (c)\ 2400 \\ \times .012 \\ \hline \end{array}$$

$$\begin{array}{r} 8\cancel{0}\cancel{0}\cancel{0} \\ \times 245 \\ \hline \end{array} \qquad \begin{array}{r} 3725 \\ \times 3\cancel{0}\cancel{0} \\ \hline \end{array} \qquad \begin{array}{r} 24\cancel{0}\cancel{0} \\ \times 1.2 \\ \hline \end{array}$$

The original examples (*a*), (*b*), and (*c*) are rewritten with the ciphers canceled and the decimal points placed in the new position or omitted.

Examples similar to the foregoing may be given as blackboard exercises.

$$\begin{array}{r} \underline{.03125}\ \textit{Ans.} \\ 32)1.00 \\ \underline{96} \\ 40 \\ \underline{32} \\ \text{etc.} \end{array}$$

Division of decimals is generally begun with integral divisors, the pupil writing the decimal point in the quotient when it is reached in the dividend, prefixing ciphers when necessary. There is no need of writing the decimal point in the partial dividends and products, nor should the pupil consider the place value of the figures in divisor, dividend, or quotient.

$$\begin{array}{r} \underline{.03125}\ \textit{Ans.} \\ 32)1.00 \\ \underline{.96} \\ .040 \\ \underline{.032} \\ .0080 \\ \underline{.0064} \\ .00160 \\ \underline{.00160} \end{array}$$

Integral divisors terminating in one or more ciphers should sometimes be employed. In these, the ciphers in the divisor are canceled, and a corresponding change is made in the dividend. In (*a*) the divisor 300 is changed to 3, which divides it by 100; the dividend is also divided by 100 by pointing off two decimals. In (*b*) the dividend has two decimals originally; the new dividend, therefore, has four places. Some teachers do not rewrite the divisor and the dividend, canceling instead the original decimal point, and rewriting it the proper number of places to the left, corresponding with the number of ciphers canceled in the divisor.

(*a*) 300)25686

$$\begin{array}{r} 3)\underline{256.86} \\ 85.62 \end{array}$$

(*b*) 300)256.86

$$\begin{array}{r} 3)\underline{2.5686} \\ .8562\ \textit{Ans.} \end{array}$$

(*c*) $3.\cancel{0}\cancel{0})\underline{2.56\cancel{.}86}$

$$\begin{array}{r} .8562 \end{array}$$

Examples of this kind are preceded, of course, by sight exercises in dividing an integer or a mixed decimal by 10, 100, etc.

When the divisor contains a decimal, the customary procedure now is to change it to an integer and to make a corresponding change in the dividend. Thus, the following examples:

(*a*) 2.7)14.58 (*b*) .027)1.458 (*c*) .27).1458

are changed as follows:

$$\begin{array}{r} 5.4\ \textit{Ans.} \\ 27\overline{)145.8} \\ \underline{135} \\ 108 \\ \underline{108} \end{array} \qquad \begin{array}{r} 54\ \textit{Ans.} \\ 27\overline{)1458} \\ \underline{135} \\ 108 \\ \underline{108} \end{array} \qquad \begin{array}{r} .54\ \textit{Ans.} \\ 27\overline{)14.58} \\ \underline{13\,5} \\ 108 \\ \underline{108} \end{array}$$

In (*a*) the divisor and the dividend are both multiplied by 10, in (*b*) by 1000, and in (*c*) by 100.

The sight and the blackboard exercises recommended in multiplication form the basis for corresponding division exercises, as, for instance:

(*a*) $.25)\underline{31}$ (*b*) $.125)\underline{31}$ (*c*) $.0625)\underline{11}$

(*d*) $2.5)\underline{31}$ (*e*) $1.25)\underline{31}$ (*f*) $6.25)\underline{11}$

These need not be carried to the same extent as the former, being less useful.

Quotients in written exercises should seldom be carried beyond four decimal places. Pupils should note the last remainder, and if the next figure is 5 or over, the fourth decimal figure should be increased by 1 followed by a minus sign. When the next figure is less than 5, a plus sign is written after the last quotient figure.

The quotients of (*a*) 18.374 ÷ 24.6 and (*b*) 18.375 ÷ 24.6 are written as follows:

```
             .7469                      .7469
(a) 246.)183.74            (b) 246.)183.75
         1722                       1722
          1154                       1155
           984                        984
           1700                       1710
           1476                       1476
            2240                       2340
            2214                       2214
              26                        126
.7469 + Ans.                          .7470 − Ans.
```

In (*a*) 18.374 is divided by 24.6, the divisor being changed to an integer, etc. The 26 remaining after the fourth quotient figure shows that the next figure would be 1; the result is therefore given as .7469 +, the sign indicating also that the division is inexact. In (*b*), the remainder 126 being greater than the half of the divisor, 246, the next quotient figure will be at least 5; the quotient is, therefore, increased by 1, and it is written .7470 −, the minus sign indicating that the result is somewhat less.

The teacher should note that the cipher in the foregoing, .7470 −, is necessary, this result being different from .747 −. The latter may mean anything from .7465 to .74699, while the value of the former is between .74695 and .74699.

Indicated multiplications and divisions containing decimals should generally be rewritten with the divisors changed to whole numbers, etc. Thus the divisor in (*a*) contains four decimal places, and the same number of places is rejected among the numbers above the line when they are rewritten at (*b*). The remaining decimal places above the

$$(a)\quad \frac{37.5 \times 18.64 \times .75}{.24 \times 62.5 \times .4}$$

$$(b)\quad \frac{37.5 \times 1864 \times 75}{24 \times 625 \times 4}$$

line, when more than one, should be contained in a single number, in order to diminish the likelihood of a mistake in canceling.

Problems in decimals should be as practical as possible. The equivalents in our weights and measures of the more common metric units furnish some types; such as the kilo of 2.2046 pounds, the meter of 39.37 inches, etc.; also the value of the German mark, 23.8 cents, of the French franc, 19.3 cents, of the pound sterling, $4.8665, etc.

At convenient times throughout the term the matter in the text-book may be read and discussed. No time should be given to the memorizing of rules or definitions.

Some further suggestions will be found in the chapter on Relation of Numbers.

CHAPTER V

COMPOUND NUMBERS

WORK in denominate numbers was formerly limited to the second half of the sixth year; but instruction in this branch with simple and compound units now forms a feature of every grade.

As the practical value of this topic is very slight, the greater part of the work should consist of oral and sight exercises, and should be limited to the more commonly used units. No example should contain more than two different units before the sixth year, and none thereafter should contain more than three. These units should not be square or cubic inches, cord feet, or the like, in exercises in reduction, addition, etc.

The method of working each form of example should be developed by the pupil under the teacher's guidance, no formal explanation being really necessary at any stage, the teacher's task being merely the preparation of suitable examples and the giving of timely hints as to the proper arrangement of the work.

Such exercises as the reduction of gallons and quarts to quarts, and of quarts to gallons and quarts, should form a portion of the two-step problems of the lower grades, which the children solve without formal instruction. The introduction of a third unit in the sixth year may be made by such oral exercises as the following:

(*a*) How many quarts in 7 gallons 2 quarts?

(*b*) How many pints in 30 quarts?

(*c*) How many pints in 30 quarts 1 pint?

(*d*) How many pints in 7 gal. 2 qt. 1 pt.?

After a sufficient number of the foregoing have been given, a pupil may find at the blackboard the number of pints in 17 gal. 3 qt. 1 pt. The arrangement usually employed is the one found at the left, which is objected to by many teachers, since it seems to imply that a certain number of gallons is multiplied by 4 and gives a product consisting of quarts. As the pupils give no thought to the matter, it does them no harm.

$$\begin{array}{l}
17\text{ (gal.) }3\text{ (qt.) }1\text{ (pt.)}\\
\underline{4}\\
71\text{ (qt.)}\\
\underline{2}\\
143\text{ (pt.)}\\
\textit{Ans. }143\text{ pt.}
\end{array}$$

Model examples in the text-books show first the change of 17 gallons to 68 quarts and the subsequent addition of 3 quarts; but upper-grade pupils should never be permitted to waste time by taking two steps where one will suffice, and they should be expected in all such examples to "add-in" the quarts, pints, etc.

WRONG FORM

$$\begin{array}{l}
17\text{ gal. 3qt. pt.}\\
\underline{4}\\
68\text{ qt.}\\
\underline{3}\text{ ``}\\
71\text{ qt.}\\
\underline{2}\\
142\text{ pt.}\\
\underline{1}\text{ ``}\\
143\text{ pt. }\textit{Ans.}
\end{array}$$

Reduction ascending is developed in the same way, a few typical oral problems preceding written ones, such as the reduction of 143 pints to units of higher denominations.

The arrangement given herewith is also objected to by over-particular teachers, but it seems to appeal more to the average pupil than any other that has been suggested. By omitting the multipliers and divisors, which are helpful rather than essential, the foregoing reductions might take the following forms:

$$\begin{array}{l}
2\overline{)143}\text{ (pt.)}\\
\;\;4\overline{)71}\text{ (qt.) }1\text{ (pt.)}\\
\;\;\;\;\;17\text{ (gal.) }3\text{ (qt.) }1\text{ (pt.)}\\
\textit{Ans. }17\text{ gal. }3\text{ qt. }1\text{ pt.}
\end{array}$$

(*a*) 17 gal. 3 qt. 1 pt.
= 71 qt. 1 pt.
= 143 pt. *Ans.*

(*b*) 143 pt.
= 71 qt. 1 pt.
= 17 gal. 3 qt. 1 pt. *Ans.*

Sight examples involving addition, subtraction, multiplication, and division of compound numbers containing two units are readily answered by pupils of the fifth year, without preliminary explanation of any kind. The following are the simpler forms:

$$\begin{array}{lll} (a)\ \text{5 lb. 8 oz.} & (b)\ \text{11 lb.} & (c)\ \text{5 lb. 8 oz.} \\ \underline{+\ \text{5 lb. 8 oz.}} & \underline{-\ \text{5 lb. 8 oz.}} & \underline{\quad\times 2\quad} \end{array}$$

(*d*) $2\overline{)11}$ lb. (*e*) 5 lb. 8 oz.$\overline{)11}$ lb.

These may be followed by such as:

$$\begin{array}{lll} (f)\ \text{5 lb. 10 oz.} & (g)\ \text{11 lb. 1 oz.} & (h)\ \text{5 lb. 8 oz.} \\ \underline{+\ \text{4 lb. 8 oz.}} & \underline{-\ \text{5 lb. 8 oz.}} & \underline{\quad\times 3\quad} \end{array}$$

(*i*) $3\overline{)\text{17 lb. 4 oz.}}$ (*j*) 1 lb. 4 oz.$\overline{)\text{7 lb. 8 oz.}}$

When the course of study calls for three different denominate units in work of this kind, an example comprising three or four addends is worked upon the blackboard without difficulty by any pupil who has had experience with the foregoing sight exercises.

As a preliminary to subtraction, successive pupils may be sent to the blackboard to fill up the blanks in examples similar to the following:

$$\begin{array}{ll} (k)\ \text{3 bu. 3 pk. 5 qt.} & (l)\ \text{16 yd. 2 ft. 9 in.} \\ \underline{+\ ?\qquad\qquad\qquad} & \underline{+\ ?\qquad\qquad\qquad} \\ \text{8 bu. 1 pk. 1 qt.} & \text{23 yd. 1 ft. 4 in.} \end{array}$$

In finding the results, the pupil will probably employ the Austrian method. In working a regular subtraction example, he will naturally follow the method he uses in subtraction of fractions. The results should, of course, be tested to ascertain whether or not they are correct. The subtraction examples corresponding to the foregoing are:

$$\begin{array}{r} (m)\ 8\text{ bu. }1\text{ pk. }1\text{ qt.} \\ \underline{-\ 3\text{ bu. }3\text{ pk. }5\text{ qt.}} \end{array} \qquad \begin{array}{r} (n)\ 23\text{ yd. }1\text{ ft. }4\text{ in.} \\ \underline{-\ 16\text{ yd. }2\text{ ft. }9\text{ in.}} \end{array}$$

Too much time should not be spent on subtraction.

The finding of the difference in time between two dates should employ the method given in the text-book. This topic is discussed again in Part Four, in the chapter on Interest.

Examples in multiplication of compound numbers may be given in connection with the addition examples, the processes being similar in requiring that a product susceptible of reduction to higher terms should be so reduced, etc.

$$\begin{array}{r} 3\text{ yd. }2\text{ ft. }8\text{ in.} \\ \times\ 5 \\ \hline 5)\underline{19\text{ yd. }1\text{ ft. }4\text{ in.}} \\ 3\text{ yd. }2\text{ ft. }8\text{ in.} \end{array}$$

As a preliminary to an example in division, one in multiplication may be worked upon the blackboard, and the product divided by the multiplier, the pupil announcing the successive steps in each operation.

Most of the exercises should be worked at sight from the text-book, the answers being written on a narrow sheet of paper, which is handed in at the end of the recitation. The pupil should not be permitted the use of any other paper on which to write superfluous figures.

As a form of division exercise the teacher may give an example asking, for instance, for $\frac{1}{5}$ of 19 yd. 1 ft. 4 in.; also an example requiring the finding, for instance, of $\frac{3}{5}$ of 19 yd. 1 ft. 4 in.

The treatment of such examples as 19 yd. 1 ft. 4 in. ÷ 3 yd. 2 ft. 8 in. will be discussed in the next chapter, as well as the reduction of the decimal or fraction of one denomination to lower units, with the reverse type.

The teacher should select from the problems of the text-book those adapted to the conditions of the neighborhood, and supplement them by others prepared by himself or brought in by the pupils.

CHAPTER VI

RELATION OF NUMBERS

There are three related types of problems which should be made a part of the work of every grade of the elementary school. They are usually stated as follows:

1. Given the whole to find a part.
2. Given a part to find the whole.
3. Finding what part one number is of another.

In the lowest grades the foregoing might be expressed in problems of these forms:

(*a*) "What is the cost of one-half of a 10-cent pie?"

(*b*) "If $\frac{1}{4}$ pound of candy costs 5 cents, what is the price of 1 pound?"

(*c*) "What part of a yard of 8-cent muslin can be bought for 4 cents?"

The young pupils give the correct answers to these without disturbing themselves as to the procedure.

As the denominate units are introduced, they are used in problems of two of the types:

(*a*) "How many inches in $\frac{3}{4}$ foot?"

(*b*) "What part of a foot is 8 inches?"

To these the answers are also readily given, but without any special consideration of the process.

Somewhere about the fifth year the pupil should be led to understand the reason for the method he employs in ascertaining, for instance, the cost of a yard of silk when $\frac{2}{3}$ yard costs 36 cents.

In a previous chapter it has been suggested that the pupil should employ in his solution of examples of this type the same method that he would use in finding the cost of a yard when $1\frac{2}{3}$ yards cost 90 cents.

Much of the poor work in percentage in the seventh year is due to the failure of the pupils to understand their method of obtaining the answers to similar problems in the fifth year when the fractional equivalent of the per cent is used.

Owing to the stress laid upon the *part* and the *whole*, the fractions given in the second type are always proper fractions, and the problem is generally solved by analysis in the manner previously explained. Since the expected answer to the third type is a proper fraction, the larger number is always made the divisor. In the fifth and sixth years, the wording "what part (or what decimal?) is 12 of 15?" always calls for a result obtained by reducing $\frac{12}{15}$ to $\frac{4}{5}$ or .8. When the percentage example "What per cent of 12 is 15?" is asked, a pupil without reflection frequently makes 15 the divisor, and gives the result as 80%, instead of the correct one, 125%.

The foregoing types should be expressed as follows:

1. Given the unit, to find the fraction (proper or improper).

2. Given the fraction (proper or improper), to find the unit.

3. Finding what fraction (proper or improper) one number is of another.

The introductory examples of the second type should employ a mixed number instead of a fraction, as follows:

(*a*) "Find the cost of a pound of butter when $1\frac{3}{4}$ pounds cost 56 cents."

A later example would use the improper fraction.

(*b*) "What is butter worth per pound when $\frac{7}{4}$ lb. cost 56 cents?"

In solving (*a*) the pupil would naturally seek the quotient of 56 divided by $1\frac{3}{4}$, and he should use the same method in solving (*b*) and the next in order, (*c*).

(*c*) "What is the price per pound when 24 cents is paid for $\frac{3}{4}$ pound?"

As has been already stated, the method in vogue of obtaining the answer to the last by first finding the cost of $\frac{1}{4}$ lb. by dividing 24 cents by 3, and multiplying this quotient by 4 to ascertain the price per pound, obscures the fact that the required process is a single-step one in division, consisting of the dividing of 24 cents by $\frac{3}{4}$, which is performed by multiplying it by $\frac{4}{3}$.

After the pupils fully understand this procedure, the connection between (*c*) and the following should be shown:

(*d*) "When $\frac{3}{4}$ of a number is 24, what is the number?"

From (*c*) they can understand that the required price per pound multiplied by $\frac{3}{4}$ gives 24 cents as the product. As they know that "of" is a sign of multiplication they may be able to appreciate the relation between (*c*) and (*d*).

The fact that division is required to solve the foregoing problems may be understood more readily by some pupils when they are written in the equation form as follows:

$$\begin{aligned}
2 \text{ times } ? \text{ cents} &= 72 \text{ cents}\\
1\tfrac{3}{4} \text{ times } ? \text{ cents} &= 56 \text{ cents}\\
\tfrac{7}{4} \text{ times } ? \text{ cents} &= 56 \text{ cents}\\
\tfrac{3}{4} \text{ times } ? \text{ cents} &= 24 \text{ cents}\\
\tfrac{3}{4} \text{ times } ? &= 24
\end{aligned}$$

Many examples of the third type should be given in which the answer is an improper fraction, such as:

(*e*) "What fraction of a foot is 15 inches?" to which the reply is $\frac{5}{4}$, obtained by reducing $\frac{15}{12}$.

The wording of problems of this type should be changed from time to time to prevent pupils from assuming that the first (or the second) is always the divisor; as, for example:

(*f*) "What fraction of 15 is 24?"

(*g*) "24 is what fraction of 15?"

The teacher will, of course, defer problems involving division by a mixed number until pupils have had considerable practice in mixed numbers. By this time, examples such as (*f*) and (*g*) may contain fractions or mixed numbers; as, for instance:

(*h*) "What fraction of $\frac{15}{37}$ is $\frac{24}{37}$?"

(*i*) "$\frac{2}{3}$ is what fraction of $\frac{3}{4}$?"

(*j*) "What fraction of $\frac{2}{3}$ is $\frac{3}{4}$?"

In solving (*h*) the denominator, 37, may be rejected, making the problem the same as (*f*) or (*g*). In working (*i*) and (*j*) at sight the pupil may change both fractions to 12ths and reject the common denominator.

When the pupils find difficulty in solving these problems at sight or orally, they should be permitted to take the necessary time to read them carefully. The chief use of the sight work is to make pupils quick and accurate in combining numbers when the operation is indicated or clearly evident. When care is needed to determine the divisor, as in the third type, the pupil should never be hurried.

Even when they contain only abstract numbers, the foregoing are real problems, inasmuch as the pupils are required to determine the process. Only after considerable familiarity with any type of work has been obtained should the methods employed in particular oral examples be used, as these tend to obscure the process required in a general example.

During the decimal work, the same types are presented in the following form when the pupils have reached the proper stage:

1. "Find 2.5 of 12."

2. "(*a*) 30 is 25 tenths of what number? (*b*) 8 hundredths of what number is 30?"

3. "(*a*) 16 is what decimal of 20? (*b*) What decimal of 16 is 20?"

In the second type the pupil has little difficulty in seeing that the result of (*a*) is indicated by 30 ÷ 2.5 and of (*b*) by 32 ÷ .08, both being examples in division.

In connection with the foregoing, similar examples using common fractions are continued.

Examples of the second type are extended in the fifth grade to include in fractional and decimal forms such problems as the following:

(*a*) "A man had 70 sheep after increasing his original flock by $\frac{2}{5}$ of their number. How many had he at first?"

(*b*) "If a woman has $75 remaining in the bank after she has withdrawn .2 of her money, how much had she at first?"

Many simple problems of this kind are useful as a means of enabling a pupil to understand his later problems in percentage.

Examples involving multiplication and division should also be included, such as:

(*a*) "Find the value of .12 of a vessel when the cost of .8 is $6300."

The solution is first indicated by writing .8 below the line to show the cost of the entire vessel, and multiplying by .12 to show the value of the required decimal of it. The work is then rewritten with the decimal points omitted, one of the ciphers in $6300 being canceled instead of writing the numbers above the line $6300 × 1.2.

$$\frac{\$6300 \times .12}{.8}$$

$$\frac{\$630 \quad \times \ 12}{8} =$$

The same example, containing fractions, reads as follows:

(*b*) "Find the value of $\frac{3}{25}$ of a vessel when the cost of $\frac{4}{5}$ is $6300."

The cost of the whole vessel is indicated by writing $\frac{4}{5}$ below the line, to show that \$6300 is to be divided by it, and the value of $\frac{3}{25}$ is indicated by writing this fraction above the line as a multiplier of the quotient. When the work is rewritten, the divisor $\frac{4}{5}$ is inverted, 5 being placed in the numerator and 4 in the denominator, and the two terms of $\frac{3}{25}$ are written above and below the line, respectively.

$$\frac{\$6300 \times \frac{3}{25}}{\frac{4}{5}}$$

$$\frac{\$6300 \times 5 \times 3}{4 \times 25}$$

If this method is to be followed in both forms of the foregoing example, it is inadvisable to make two steps of the following:

"Find the value of a vessel when $\frac{4}{5}$ of it is worth \$6300."

The method of "unitary analysis" is intended for the solution of examples that may be solved by proportion, when three terms are given, from which the fourth is to be found. In such an example as the following:

"Find the cost of 12 cows at the rate of \$6300 for 80 cows," two operations are indicated, the first consisting of division by 80 to find the cost of 1 cow, and next the multiplication of the latter by 12.

$$\frac{\$6300 \times 12}{80}$$

A problem that involves the finding of the value of the whole when that of $\frac{4}{5}$ is given, contains only two terms, and is not one that really calls for proportion, even though it could be stated as follows:

$$\frac{4}{5} : 1 :: \$6300 : x$$

During the work in denominate numbers in the sixth year, the foregoing types may take these forms:

1. "(*a*) Change $\frac{4}{5}$ day to hours and minutes. (*b*) How many pecks and quarts in .375 bushel?"

2. "(*a*) What decimal of an hour is 7 min. 30 sec.? (*b*) What fraction of a gallon is 3 pints?"

Examples of this kind should be given incidentally and in connection with those heretofore recommended, and should not be taught as a separate subdivision of denominate numbers. That the reduction of .375 bu. to pecks follows the method employed in reducing 3 bushels should be apparent to a sixth-grade pupil. The necessity of changing the expression $\frac{\text{7 min. 30 sec.}}{\text{1 hour}}$ to $\frac{7\frac{1}{2}\text{ min.}}{\text{60 min.}}$ or to $\frac{\text{450 sec.}}{\text{3600 sec.}}$ may require some questioning to bring out the fact that the divisor and the dividend should be of the same denomination, and that in such a case the denomination may be omitted from both. After this has been brought home to the pupils, an occasional example in dividing one compound denominate number by another may be given, if it has not previously been done.

CHAPTER VII

MEASUREMENTS

By the end of the sixth year the work in measurements includes the finding of the surfaces of rectangles and of right triangles, and the surfaces and the volumes of rectangular solids, together with some practical applications.

When the pupils have become familiar with the meaning of the square inch, square foot, and square yard in the primary grades, the subdivision of a rectangle into square units (or unit squares) should no longer be necessary. The employment of rough diagrams in which the dimensions are written should, however, be encouraged.

In the early problems that require the area in square feet, square yards, etc., the pupils should express the dimensions in linear units of the corresponding kind, making the necessary reductions. The examples given by the teacher should be of such a character as not to call for two fractional factors until the completion of multiplication of fractions in the fifth year.

(*a*) 36 in. × 36 in.
(*b*) 3 ft. × 36 in.
(*c*) 1 yd. × 3 ft.
(*d*) 1 yd. × 36 in.
(*e*) 3 ft. × 3 ft.
(*f*) 1 yd. × 1 yd.

Examples requiring the area in square inches may be given in the accompanying forms. In each of the last five, the reduction to inches of the dimensions expressed in feet or yards should first be made. Early examples in finding areas in square feet should have only one dimension that is expressed as a fraction of a foot. The fact that a square foot contains 144

(*a*) 6 ft. by 1½ ft.
(*b*) 6 ft. by 18 in.
(*c*) 2 yd. by 1½ ft.
(*d*) 2 yd. by ½ yd.

square inches need not be learned at once. When areas are required in acres the dimensions should be expressed in rods when convenient. In ascertaining the result, the dimensions in rods may be written above a line, and the number of square rods to an acre, 160, written as a divisor, as in the accompanying example, which requires the area in acres of a field 48 rods long and 25 rods wide. When the dimensions are given in yards, the area in acres should be obtained by writing 4840 as the divisor, and shortening the work as much as possible by cancellation.

A. $\frac{48 \times 25}{160}$

The teacher should not dwell unnecessarily upon the impossibility of multiplying feet by feet, etc. It is safer not to allude to it at all, especially as the pupils may be taught later to do so. English books, as a rule, compiled by excellent mathematicians, state that feet multiplied by feet give square feet, and that square feet multiplied by feet give cubic feet. While this may not be strictly correct, they employ that method instead of using our circuitous way: 1 sq. ft. × 12 × 16 = 192 sq. ft.

$$\begin{array}{rr} & 16 \text{ ft.} \\ & \times\ 12 \text{ ft.} \\ \hline \textit{Ans.} & 192 \text{ sq. ft.} \end{array}$$

As much as possible of the work in measurements should be the finding of the areas of rectangles in the vicinity of the school. The dimensions of the floor and of the walls of the class-room should be obtained by the pupils and the surfaces calculated, as well as the number of square inches of glass in the windows, the area covered by the school building, the area of the school enclosure, etc.

Sight exercises in finding the areas of rectangles may be employed as a means of reviewing some of the short methods in multiplication already learned, such as:

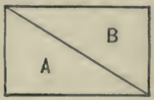
84×25 $\quad$ 73×99 $\quad$ $32 \times 19\frac{7}{8}$

etc. $\quad$ etc. $\quad$ etc.

The foregoing, and other similar combinations, may also be concealed under the following forms, in which the area in square inches is required of rectangles having dimensions as follows:

7 ft. long, 2 ft. 1 in. wide
8 ft. 3 in. long, 6 ft. 1 in. wide
2 ft. 8 in. long, 1 ft. $7\frac{7}{8}$ in. wide

It sometimes arouses a class to discover that a task seemingly very difficult is not beyond their powers.

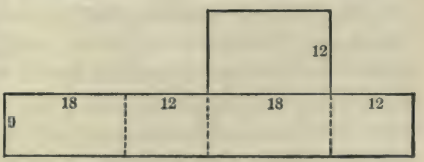


The fact that the area of a right triangle is one-half the product of its dimensions should be discovered by the pupils from a diagram in which a rectangle is divided into two equivalent right triangles. In giving the answers to sight examples the pupils should employ the half of the dimension that will yield the more convenient multiplier. Thus:

$$
\begin{aligned}
(a)\ \tfrac{1}{2}\,(16 \times 19) &= 8 \times 19\\
(b)\ \tfrac{1}{2}\,(24 \times 32) &= 12 \times 32\\
(c)\ \tfrac{1}{2}\,(44 \times 22) &= 44 \times 11
\end{aligned}
$$

In (*a*), $\frac{1}{2}$ of 16 is used, since 19 is odd; in (*b*), the product of 12 and 32 is obtained more readily than that of 24 and 16; in (*c*), 11 times 44 is simpler than 22 times 22.

The surfaces of rectangular solids are taught chiefly in connection with floors, walls, etc., for the purpose of ascertaining the cost of painting, plastering, carpeting, etc. In order that pupils may understand the method of finding a convex surface, such as that of the walls of a room, the diagram of the room should be drawn, and the fact obtained



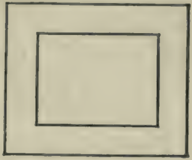
by means of questions that the area of the four walls in the accompanying diagram is the perimeter of the floor, 60 ft. (18 ft. + 12 ft. + 18 ft. + 12 ft.), multiplied by the height. For plastering purposes the area of the ceiling is included. If the area is required in square yards, the preliminary reductions may be made.

The teacher should omit all so-called practical measurements found in the text-books except such as are prescribed in the course of study. In working these the pupils should make inquiries as to the methods in vogue among the mechanics in the neighborhood. As a rule, the answers obtained by young children are of very doubtful accuracy. The latter are unable to appreciate the allowances that must be made in ascertaining the cost of doing the same amount of work under different conditions.

The rule, as given in some of the books, states that in finding the quantity of material used, full deductions are made for openings, but that only one-half is deducted in finding the cost of labor. As a person having work done seldom or never pays for it by the square yard or cubic foot, the average pupil is wasting his time in finding a number of incorrect results by working problems requiring the expense of performing a piece of work. Exercises of this kind limited to one particular trade might have some value for a boy who later on follows this trade, but even he should not be required to assimilate a quantity of misinformation concerning a number of other callings. The time that a pupil can devote to arithmetic is spent more profitably in developing mathematical ability than in acquiring a very inadequate acquaintance with a number of unrelated things.

An example in carpeting may sometimes be made clear to a pupil by requiring him to draw the given room to a scale and with the same scale to indicate the strips of carpet. The introduction of the border with its beveled corners and the consequent waste should be postponed for a higher grade. After a diagram is drawn on the blackboard the class should be able to ascertain the number of strips by dividing the proper dimension by $\frac{3}{4}$, the width (in yards) of all carpet except ingrain.

A type of problem is frequently found which requires the area of a walk surrounding a rectangular plot. This is best obtained in this grade by deducting the area of the inner rectangle from that of the outer. Assuming the dimensions of the former to be 24 feet by 20 feet, and the width of the walk to be 4 feet, the dimensions of the outer rectangle will be 32 feet by 28 feet, twice the width of the walk being added to each. When the dimensions of the outer rectangle are given, twice the width of the walk is deducted from each to obtain the dimensions of the inner one.



In determining the work in calculating volumes, the teacher must be guided by the course of study. These frequently require the finding of the capacity of bins, tanks, etc. Examples of this kind should be worked by cancellation, employing the bushel of 2150.4 cubic inches and the gallon of 231 inches, unless the text-book answers require the employment of $7\frac{1}{2}$ gallons to the cubic foot and $1\frac{1}{4}$ cubic feet to the bushel. In this case the pupils may solve the problems asking (*a*) the number of gallons in a cubic foot when a gallon contains 231 cubic inches, and (*b*) the number of cubic feet in a bushel of 2150.4 cubic inches. The answer to each carried out to two decimal places may then be compared with the equivalents given in the text-book.

When the latter states the capacity of each in cubic inches, the pupils may employ the foregoing problems to obtain the approximate equivalents for use in sight work.

The legal capacity of the bushel, 2150.42 cubic inches, is generally replaced by the one given on page 225 — 2150.4 cubic inches — as the latter may be canceled by 8 several times, and also by 3 and by 7.

When the dimensions of a bin are given in feet, as, for instance, 48 ft. by 35 ft. by 8 ft., the reduction of each to inches may be indicated by writing 12 after it preceded by the sign of multiplication.

$$\frac{35 \times 12 \times 48 \times 12 \times 8 \times 120}{21504}$$

In the example given herewith a cipher is annexed to the last number (12) to compensate for the decimal point omitted in the divisor.

The pupils should verify for themselves the contents of a gallon by pouring a gallon or a quart of sand into a box and measuring the dimensions of the sand after leveling it. The same should be done with a half or a quarter peck of sand, from which the contents of a bushel are calculated, and the results compared with those stated in the book.

The problems should be made as practical as possible, such as the finding of the capacity of a cart or a wagon body, a corn crib, a freight car filled to a given height, etc. Pupils should be encouraged to bring in as many different exercises of this kind as they can procure in the neighborhood.

The weight of a cubic foot of water, 1000 ounces, should be employed in occasional problems requiring the weight of a wooden beam, of a block of granite, of a cake of ice, etc., the dimensions being given, together with the specific gravity of the wood, granite, etc.

The sight exercises should include some types that require the pupil to select the proper order of combining the numbers. The following are examples:

$12\frac{1}{2} \times 47 \times 8$	(100×47)
$11 \times 59 \times 9$	(59×99)
$19\frac{3}{4} \times 8 \times 4$	$(32 \times 19\frac{3}{4})$
$47 \times 6 \times 1\frac{2}{3}$	(47×10)

Blackboard exercises of the same types are such as:

$12\frac{1}{2} \times 647 \times 2$	(647×25)
$385 \times 6 \times 1\frac{1}{3}$	(385×8)
$243 \times 4\frac{1}{2} \times 2$	(243×9)
$684 \times 5 \times 8$	(684×40)

CHAPTER VIII

PERCENTAGE AND INTEREST

PERCENTAGE, which was formerly postponed until the seventh year, is now generally taught in a simple way in the sixth year. After the pupils have been informed that "per cent" means "hundredths," and that 7 per cent indicates 7 hundredths, they should be able to find 7 per cent of $300, 6 per cent of 200 acres, etc. If a pupil who is asked to calculate 4% of $273 at the blackboard hesitates about the arrangement of the numbers, he is easily led to see that the process is merely the multiplication of $273 by 4 hundredths, the latter being written in the decimal form. The words *base*, *rate*, and *percentage* may be introduced as occasion requires, but without any formal definitions at this stage.

$$\begin{array}{r} \$273 \\ \times\ .04 \\ \hline \$10.92\ \textit{Ans.} \end{array}$$

$$\begin{array}{r} 368 \text{ cows} \\ \times\ .25 \phantom{\text{ cows}} \\ \hline 92 \text{ cows} \end{array}$$

An example involving the finding of 25 per cent of 368 cows should perhaps be written with 25 hundredths as the multiplier, but the result should be obtained by multiplying 368 cows by $\frac{1}{4}$, the equivalent fraction.

In his treatment of percentage the teacher should constantly bear in mind that no new process which requires special development is under consideration. The pupils will find no difficulty in understanding the examples if the teacher is careful to refrain from explaining that "the percentage is found by multiplying the base by the rate expressed as hundredths." The pupils, already fully aware that .25 of 368 means the product of these two factors, are likely to assume that there is something mysterious in a topic that requires a rule to tell them to do what they would naturally do without special direction.

Unless required by the course of study, the teacher should confine his work in percentage to the so-called first case. Although the other cases present no particular difficulty, their treatment requires time, which is not always readily obtainable. Examples of these general types can be given in the fraction and decimal reviews, which will furnish the same mental discipline. The methods of teaching these cases of percentage will be treated in a subsequent chapter.

The problems in sixth year percentage should be self-explanatory. There should be no time devoted to the terminology of commission, insurance, etc., in this grade, the wording of each problem furnishing sufficient information for present purposes. The following may suggest appropriate forms:

(*a*) "An agent sold a farm for \$3375, receiving 3% of this sum as commission for his services. How much was his commission?"

(*b*) "Mr. Johnson insured his house for \$2500, at a cost of 1¼% of this sum. How much did the insurance cost him?"

(*c*) "A merchant imported from Europe goods valued at \$8750, on which the government charged him duty at the rate of 45%. How much was the duty?"

The foregoing represent a few of the special types that the teacher may desire to give. The work may include many others that contain no unfamiliar terms.

(*d*) "A farmer bought a cow for \$45 and sold it at an advance of 25%. What did he receive for the cow?"

(*e*) "Goods marked at \$450 were sold at a reduction of 12½%. Find the selling price."

The sight exercises should contain some examples in the reduction of fractions to decimals and to per cents, and vice versa.

Other exercises that require the finding of the percentage should be of two general forms. Those that are given as a preliminary to written work should be largely of the type that requires the multiplication of the base by a rate that is not changed to a common fraction; such as:

(*a*) 3% of 200
(*b*) 4% of 150
(*c*) 5% of 700
(*d*) 6% of 450
(*e*) 7% of 300
(*f*) 8% of 250

The following, which are worked by reducing the per cents to fractions, should be postponed until the pupils understand that percentage is obtained by multiplication. When they find 25% of 48 by dividing the latter by 4, they are apt to forget that they have multiplied it by $\frac{1}{4}$.

(*g*) 25 % of 48
(*h*) $33\frac{1}{3}$% of 96
(*i*) $12\frac{1}{2}$% of 88
(*j*) $6\frac{1}{4}$% of 32
(*k*) $16\frac{2}{3}$% of 66
(*l*) 50 % of 64

The teaching of interest in the sixth year should be as simple as the course of study will permit. The first examples should require the interest for 1 year at a given per cent. This may be given in the form of a percentage problem without preliminary explanation, as follows:

(*a*) "A woman has $375 in the savings bank, on which she receives 4% interest each year. How much is the annual interest?"

(*b*) "A man borrows $375, for the use of which he agrees to pay 4% interest each year. How much interest does he pay annually?"

A few drill exercises may be given in finding the interest for 2 years, $2\frac{1}{2}$ years, 6 months, etc., using only small fractions of the year, or aliquot parts of the year expressed in months.

As a rule, these should be worked by finding the interest for 1 year, then multiplying this by the number of years. The form given in the accompanying examples is recommended for sixth year pupils. In these, the interest for 1 year is denoted by the statement, as is the interest for 2 years 6 months and for 4 months. The short method of combining the two factors in (*a*) into one of 10% should be postponed for the present, interest for any period other than 1 year being obtained in two steps, to emphasize the fact that interest for a given time is obtained by multiplying the interest for 1 year by the number of years.

	(*a*) $375
	.04
Int. for 1 yr.	$15.00
	2½
	7.50
	30.
Int. for 2 yr. 6 mo.	$37.50 *Ans.*

	(*b*) $375
	.04
Int. for 1 yr.	$15.00
	× ⅓
Int. for 4 mo.	$ 5.00

The teacher should remember that percentage and interest are not new processes; they are simply names for topics that are worked by methods already familiar. The space devoted to interest in the text-book and the time spent in working a large number of examples tend to lead the teacher to the belief that this topic has considerable importance, which is not the case. In the few callings that require many interest calculations, tables are used. The average individual is seldom called upon to find interest for years, months, and days.

After children have become familiar with the new terms, they may be encouraged to employ such short methods as are readily understood by them, but chiefly for the development of their number sense and not because of the importance of the topic. In order, therefore, to give them in the short time that can be devoted to percentage and interest in the sixth year as great an acquaintance as possible with underlying principles, pupils should not be confused by anything that will tend to prevent them from obtaining this result.

FOURTH PART — SEVENTH AND EIGHTH YEARS

CHAPTER I

REVIEW EXERCISES

The study of arithmetic in the elementary school should not close without some insight into the reasons underlying the various processes, together with a review of the short methods already taught and instruction in a few others.

The simplicity of the Arabic system of naming and writing numbers may be brought out by means of questions. The few names that with their suffixes, etc., are sufficient to indicate any number, however large, can be elicited from the pupils. It should be brought out that after the first twelve names the numbers to ninety-nine employ only the suffix "teen" or "ty" with the first nine names or with slight modifications thereof, that following these come only the new words: hundred, thousand, million, etc.; fewer than twenty words in all and two suffixes.

A comparison of the Arabic with the Roman system may be made, to show how much more readily the former lends itself to the expression of a number of any size, and the close relation between numeration and notation. Except in the case of the numbers from 11 to 19, inclusive, the figures used in writing a number in the Arabic system follow the order in which it is expressed in English. If the pupils are acquainted with foreign languages, the different methods employed to express 18, for instance, in Latin, French, and German, respectively, may be alluded to; also those used in English, German, and French, respectively, to express 69 and 96, for instance. The place value of figures in the

Arabic system, as compared with that of the letters employed in the Roman notation, may be taken up under the head of reduction.

The text-book definitions of the various kinds of numbers may be read in class and discussed, but the pupils derive no benefit from memorizing them. The separation of numbers into concrete and abstract should be treated, as well as the meaning of *like numbers*. The statements regarding the various forms of numbers, such as integers, fractions, decimals, with their subdivisions, should also be read from the text-book, and a few questions concerning the method of reading and writing each kind should form a part of the work done in numeration and notation at this time.

The teacher might add a word or two as to some of the abbreviations employed by business men: such as, three seventy-five, twelve ninety-two, etc., in reading 375, 1292, etc.; ten fifty-six, instead of ten dollars and fifty-six cents; the employment of the separate figures in the dictation of a mixed decimal; one, four, three, two, point, o, six, to indicate 1432.06; etc. The fact that in bills $13\frac{1}{2}$, $14\frac{3}{4}$, $27\frac{1}{4}$ are frequently written 13^2, 14^3, 27^1, without denominators, may also be mentioned, if the teacher thinks it worth while. It may be of interest to pupils to learn that the comma (,) is used in France and Germany as the decimal point, and not the period (.), these countries using the latter as a sign of multiplication, together with the one used in this country (×); also that the only sign of division in these same countries is the colon (:), which is used in the United States solely to denote ratio.

If the pupils have used letters in solving problems, the reading and the writing of an algebraic term may constitute one of the features; also the method of designating lines, angles, etc.

The special method of reading a fraction composed of large numbers, $\frac{2347}{5893}$, for instance, as "2347 over 5893" may be stated, or "numerator 2347 denominator 5893." Complex fractions are also read in both of these ways. The necessity of reading .00001 as "point 00001," or in some similar manner, to distinguish it from 100 thousandths may be discussed; as well as the mistake that may be made in writing from dictation 300.007, 200$\frac{5}{7}$, etc., which may be taken as .307, $\frac{205}{7}$, unless one is read "300 and the decimal 7 thousandths," and the other "200 and the fraction $\frac{5}{7}$." The use of the word "and" in stating numbers, solely to separate an integer from a fraction or a decimal, is not so universally accepted as to constitute a sufficient safeguard, many people still continuing to say three hundred and seven, two hundred and five to denote 307 and 205, respectively.

Reduction is next in order, the meaning of the term being first taken up; namely, that it is a change in the form of an expression without a change in its value, and the fact that the common use of the word to denote a lessening does not apply to its employment in mathematics. The examples in reduction should cover all the arithmetical topics. Pupils should be called upon to express a number, three millions, for instance, as 3 millions, 30 hundred thousands, 300 ten thousands, 3000 thousands, 30,000 hundreds, and 300,000 tens; also as 30,000,000 tenths, etc.

The value of the different figures in a given number may constitute one of the exercises, as well as the division of the number represented by one of the figures by the number represented by another; such as, "In the number 8402.05 find the quotient of the value represented by the first figure by that indicated by (*a*) the second, (*b*) the fourth, (*c*) the sixth, (*d*) the second and third, (*e*) the fifth and sixth, etc."

In the foregoing example it is necessary to announce that 8 is to be considered the first figure and 5 the sixth, as the right-hand one is generally considered the first in other connections.

The change of a fraction to lower terms and to higher terms should also form a part of this work, together with the change of a mixed number to an improper fraction and of an improper fraction to a mixed number, as well as the simplification of a compound or a complex fraction. Changes of decimals to common fractions, and the reverse, come also under this head, as do the examples classified under reduction ascending and descending of compound numbers. Pupils should be asked to suggest other forms, which might include the change of the fraction or the decimal of a day, for instance, to hours and minutes, or hours and minutes to the fraction of a day.

In all of these reductions the meaning of each unit should be developed: the unit of each of the various orders in decimals and integers; the unit of such a fraction as $\frac{4}{5}$, with the full signification of each of its terms; the different units in a compound denominate number; etc. Pupils should be able to explain each type of reduction in such a way as to show that they understand the principles involved, after which sight examples may be answered rapidly without explanations. The written exercises should be reasonably simple and limited largely to business combinations. A few examples may require the finding of the least common multiple and the greatest common divisor by the method already taught or by the one discussed elsewhere in this chapter.

With each type of example, an appropriate test of the correctness of the solution should be stated by the pupils in answer to questions put by the teacher.

The work in the several arithmetical operations should be preceded by a brief summary giving the signs, the names of the terms, etc., as follows:

Operation	Illustration	16 is Called	2 is Called	Name of Result
Addition	$16+2=18$	Addend	Addend	Sum
Subtraction	$16-2=14$	Minuend	Subtrahend	Difference or Remainder
Multiplication	$16\times2=32$	Factor or Multiplicand	Factor or Multiplier	Product
Division	$16\div2=8$	Dividend	Divisor	Quotient
Ratio	$16:2=8$	Antecedent	Consequent	Ratio
Involution	$16^2=256$	Base	Exponent	Power
Evolution	$\sqrt[2]{16}=4$	Base	Index	Root

The method of reading the combinations employed as illustrations may be spoken of, as well as the two methods of reading 16 × 2 when both numbers are abstract.

In order that the addition examples may include common fractions, the incidental operations with whole numbers may constitute the next series of exercises; including the decomposition of an integer into its prime factors, the finding of the least common multiple and the greatest common divisor by the factor method, the rejection of common factors in cancellation, the extraction of the square and the cube root by the use of factors, etc. The underlying principles should in all cases be developed by questions put to the pupils, the exercises being of simple types and carefully prepared beforehand. A few examples slightly more difficult may be given for written work.

The test of the divisibility of numbers by 2, 3, 5, and 9 should be made the topic of a few blackboard exercises, which should also be extended to include 4, 8, 25, and 125. The fact that a number divisible by 2 and by 5 is divisible by 10 should be brought out, also the test of divisibility by such numbers as 12, 15, and 45, not that these should be used in cancellation, but to illustrate the underlying principle.

The exercises in addition, subtraction, etc., should include under each operation integers, fractions, decimals, and compound numbers before proceeding to the next. Here, too, the examples should be short, the object being to teach that in addition and subtraction like numbers are combined, and to show how this rule applies to all classes of numbers. It should be shown that in multiplication the multiplier must be abstract, although the concrete factor is actually used as a multiplier in obtaining the product. In division the fact is brought out that 12 hundred divided by 3 hundred gives the same quotient as 12 fifths divided by 3 fifths, 12 ounces divided by 3 ounces, etc. The value of each of the different figures in a multiplier or a divisor is developed and the effect of each on the several figures of the product or the quotient.

The longer examples in the fundamental operations are employed in the reviews that are had throughout the course, and there should be at least one such example in each review recitation, in order to prevent pupils from losing the facility in combining numbers that they obtained in the lower grades. The long example in addition, for instance, may well include decimals, and be a review in writing numbers. The time taken by the class should be noted, to ascertain whether or not a falling-off in speed takes place from week to week.

A short example in addition of decimals will serve as a basis for a pupil's explanation of the reason for writing each order in the same column and for the reduction of the sum of any column, when it exceeds 10. A few addition examples should be placed upon the blackboard to which only the results are written; these should be arranged in lines and also in columns. Some examples combining the

$$\begin{array}{rrrl} \$36.54 + & \$\ 9.83 + & \$27.75 & = ? \\ 8.95 + & 25.15 + & 6.53 & = ? \\ 63.27 + & 84.00 + & 38.90 & = ? \\ 6.73 + & 9.82 + & 27.75 & = ? \\ \hline ?\ + & ?\ + & ? & = ? \end{array}$$

sums of lines and columns, as shown in the accompanying type, should also form a part of the review work. A few sight examples are useful to teach the method employed by accountants in adding two columns at a time. In the example given herewith, these may first be added aloud by a pupil, as follows: 107, 111, 161, 169, 199, 205. The first addend, 27, is combined with 80 of the second to make the first sum announced, 107. To this is added the ones of the second addend, 4, making the next sum, 111. The same plan is followed with each successive addend, first combining its tens with the previous total and then its ones.

$$\begin{array}{r} 36 \\ 58 \\ 84 \\ \underline{27} \end{array}$$

Pupils of the upper grades are more interested in the earlier operations when a modification of a previous method is employed, or a new type of example. The oral exercises in addition should generally be limited to two addends, as a rule, but the combinations should be gradually increased in difficulty. The following examples suggest the grading for the last two years:

7th year 1st half: (*a*) 367 + 96 (*b*) 78 + 556
2d half: (*c*) 167 + 425 (*d*) 347 + 144
8th year 1st half: (*e*) 267 + 155 (*f*) 347 + 164
2d half: (*g*) 267 + 355 (*h*) 347 + 264

The addends of the first half of the seventh year consist of a number of three figures and a number of two figures; those of the remaining three terms consist of two numbers of three figures each. In two of the terms the hundreds' figure of one addend is 1, and the combinations in the first of these terms involve no increase of the total of the hundreds' figures.

In combining these numbers the pupil omits the word hundreds, saying in the first, 367 + 96, four fifty-seven (367 + 90), four sixty-three (457 + 6). In (*c*), 167 + 425, the successive sums are: five sixty-seven (167 + 400), five eighty-seven (567 + 20), five ninety-two (587 + 5). In (*e*), 267 + 155, they are: three sixty-seven (267 + 100), four seventeen (367 + 50), four twenty-two (417 + 5). In (*g*), 267 + 355, they are: five sixty-seven (267 + 300), six seventeen (567 + 50), six twenty-two (617 + 5).

Written examples in fractions should be limited to four addends and numbers of two figures. Sight and blackboard examples should consist of only two addends, written in lines and also in columns. The written and blackboard addition of denominate numbers should contain not more than three successive units in any example, and the number of addends should be limited as in fractions. Most of the work in denominate numbers should be done orally from the blackboard, the chief use of these exercises being the reductions involved, which should be accurately stated in the explanation of a few type examples.

If literal arithmetic constitutes a part of the course, a few sight examples in adding three or four monomials, and a few in combining three or four binomials, serve to show the difference between algebraic and arithmetical addition, through the introduction of negative numbers into the former.

If pupils have not as yet been made acquainted with the so-called Austrian method of subtraction, this should be presented. Some of the subtraction examples should provide for the deduction of the sum of several numbers from one minuend in a single operation. There should also be included a few examples and problems involv-

$$\begin{array}{r} \$100.00 \\ \left\{\begin{array}{r} 16.84 \\ 8.97 \\ 24.36 \end{array}\right. \\ \hline \textit{Ans.}\ \$49.83 \end{array}$$

$$\begin{array}{rcrl} 14.7 & - & 8.33 & = ? \\ 150. & - & 64.285 & = ? \\ 14.02 & - & .007 & = ? \\ 83.6 & - & 5.05 & = ? \\ \hline ? & - & ? & = ? \end{array}$$

ing the finding of the differences of several numbers, and also the difference between the sum of the minuends and the sum of the subtrahends.

The review work in subtraction should also include short examples in fractions and denominate numbers, to show the procedure when the fraction in the subtrahend is greater than that in the minuend, and the like case in compound numbers. These may be introduced by sight exercises such as the following:

$$
\begin{array}{r}
(a)\ 3\tfrac{3}{4} \\
+\ ? \\ \hline
10
\end{array}
\qquad
\begin{array}{r}
(b)\ ? \\
+\ 5\tfrac{7}{8} \\ \hline
10\tfrac{1}{4}
\end{array}
\qquad
\begin{array}{l}
(c)\ 3\text{ lb. }12\text{ oz.} \\
+\ ? \\ \hline
10\text{ lb.}
\end{array}
\qquad
\begin{array}{r}
(d)\ ?\qquad\qquad \\
+\ 5\text{ lb. }14\text{ oz.} \\ \hline
10\text{ lb. }\ 4\text{ oz.}
\end{array}
$$

in each of which are given the sum of two numbers and one of the numbers from which the other is to be calculated. In (*a*) the pupil thinks ¾ and ¼ are 1, 4 and 6 are 10, and gives the answer, 6¼; in (*c*) he thinks 12 oz. and 4 oz. are 1 lb., 4 lb. and 6 lb. are 10 lb., and gives the answer 6 lb. 4 oz. In (*b*) he thinks ⅞ and ⅜ are 1¼, 6 and 4 are 10, and gives the answer, 4⅜; in (*d*) he thinks 14 oz. and 6 oz. are 1 lb. 4 oz., 6 lb. and 4 lb. are 10 lb., and gives the answer, 4 lb. 6 oz.

Written exercises may be used to review addition, as well as to give practice in subtraction. The numbers should be no larger than the following:

$$
(a)\ \begin{array}{rcrcl}
42\tfrac{1}{2} & - & 16\tfrac{2}{3} & = & ? \\
30\tfrac{1}{3} & - & 25\tfrac{3}{4} & = & ? \\
27\tfrac{1}{4} & - & 7\tfrac{2}{3} & = & ? \\
6\tfrac{1}{5} & - & \tfrac{3}{8} & = & ? \\ \hline
? & - & ? & = & ?
\end{array}
\qquad
(b)\ \begin{array}{lcrcl}
16\text{ bu. }2\text{ pk.} & - & 8\text{ bu. }3\text{ pk.} & = & ? \\
24\text{ bu.} & - & 16\text{ bu. }2\text{ pk.} & = & ? \\
18\text{ bu. }1\text{ pk.} & - & 5\text{ bu. }3\text{ pk.} & = & ? \\
7\text{ bu. }2\text{ pk.} & - & 3\text{ pk.} & = & ? \\ \hline
? & - & ? & = & ?
\end{array}
$$

In the ordinary subtraction exercises in denominate numbers, not more than three successive denominate units should be used in any example. Those employed should be the ones that occur in business transactions, such as English money, time, and occasionally troy weight. If any of these tables have not formed part of the previous work, the pupils should be permitted to use their text-books to obtain the respective values of the units. No exercise should contain different units of square or of cubic measure.

There should be a few simple examples in finding the number of degrees of latitude between two places situated on opposite sides of the equator, to bring to the attention of the pupils that addition is sometimes employed to find certain differences. The difference in longitude between two places on different sides of a prime meridian should also be required.

The oral problems may include combinations of integers similar to those recommended for addition, one of the two numbers used in the addition problems being taken as the subtrahend or the remainder, and their sum being used as the minuend:

(*a*) 266 − 89
(*b*) 266 − 189
(*c*) 210 − 75
(*d*) 366 − 189
(*e*) 764 − 689
(*f*) 453 − 125
(*g*) 386 − 258
(*h*) 386 − 128
(*i*) 492 − 394

By diminishing 266 first by 80 (186), then by 9, the result, 177, is obtained in the most direct way.

Comparatively few oral exercises in subtraction of fractions or of denominate numbers should be given.

As an oral exercise in addition, combined with subtraction by the Austrian method, an example may be written on the blackboard by a pupil, who writes each figure of the result as it is announced by successive pupils at their seats. The first says aloud: 10, 13, 15, 21, and 9, are 30, and the pupil at the blackboard writes 9, the number spoken after "and." The next pupil, carrying 3, says 12, 17, 26, 31, 39, 43, and 7, are 50. The 7 is written and the next pupil carries 5. Even

$$\begin{array}{r} \$1000.00 \\ \hline 83.46 \\ 259.82 \\ 77.53 \\ 164.92 \\ 3.50 \\ 18.00 \\ 9.98 \\ \hline \text{etc. } .79 \end{array}$$

longer examples should occasionally be employed. The result in each should be tested by adding the columns downwards and including the remainder, one pupil taking each column.

The review exercises in multiplication should teach the pupils that a beginning may be made with any figure of the multiplier. Thus the product of 476 (*a*) by 841, (*b*) by 148, and (*c*) by 418 may be obtained by using the multiplicand in each case as one of the partial products:

(*a*) 476 × 841	(*b*) 476 × 148	(*c*) 476 × 418
1904	1904	1904
3808	3808	3808
400316 *Ans.*	70448 *Ans.*	198968 *Ans.*

In (*a*) the first multiplier is 1 one, the second is 4 tens, and the third is 8 hundreds; in (*b*) the first multiplier is 1 hundred, the second is 4 tens, and the third is 8 ones; in (*c*) the first multiplier is 1 ten, the second is 4 hundreds, and the third is 8 ones, the partial products being arranged accordingly. In multiplying by 8, twice the product by 4 is taken, which makes it advisable to use the latter as a multiplier before the former.

In each of the foregoing results the pupil should state the reason for the number of figures in the product, which constitutes one check. The test by casting out 9's and 11's will be discussed later.

Another type of multiplier consists of such numbers as (*a*) 328 and (*b*) 832, in which 32 is a multiple of 8. To multiply 476 by either of these, only two partial products are necessary.

(*a*) 476
× 328
3808 (product by 8)
15232 (product by 320 = 40 times product by 8)
Ans. 156128 (product by 328)

(*b*) 476	
832	
3808	(product by 800)
15232	(product by 32 = 4 times product by 8)
Ans. 396032	(product by 832)

In (*a*) the second partial product is that by 32 tens, and its right-hand figure belongs in the tens' place; in (*b*), the second partial product being that of 32 ones, its right-hand figure belongs in the ones' place. The right-hand figure in each case is written under the right-hand figure of the number producing it when the latter consists of more than one order. The right-hand figure of the product by 8 comes under 8 in both instances, occupying the ones' place in (*a*) and the hundreds' place in (*b*).

(*a*) 476	
× $42\frac{2}{3}$	
952	product by 2
$317\frac{1}{3}$	" " $\frac{2}{3}$
1904	" " 4 tens
Ans. $20309\frac{1}{3}$	product by $42\frac{2}{3}$

(*b*) 476	
× $24\frac{2}{3}$	
952	product by 2 tens
$317\frac{1}{3}$	" " $\frac{2}{3}$
1904	" " 4
Ans. $11741\frac{1}{3}$	product by $24\frac{2}{3}$

In multiplying 476 (*a*) by $42\frac{2}{3}$ or (*b*) by $24\frac{2}{3}$, the product by 2 should precede the product by $\frac{2}{3}$, the latter being obtained by dividing the former by 3. In (*b*) the product of $\frac{2}{3}$ is obtained by dividing the product of 2 tens by 3, and writing the first quotient figure one place to the right. The right-hand figure of the product by 4 is placed in the ones' column; it is twice the product by 2.

The employment of such multipliers as $\frac{7}{8}$, $9\frac{5}{6}$, $19\frac{4}{5}$, $29\frac{3}{4}$, etc., should be continued in sight and written exercises to teach the pupils that business men find it easier to multiply by $10 - \frac{1}{6}$ than by $9 + \frac{5}{6}$, etc. In finding $\frac{7}{8}$ of 99 without a pencil, a business man deducts from 99 one-eighth of

99, or $12\frac{3}{8}$, thereby obtaining $86\frac{5}{8}$, considering the subtraction less difficult than the multiplication of $12\frac{3}{8}$ by 7.

(*a*)
$$\begin{array}{r} 493 \\ \times .25 \\ \hline 123.25 \textit{ Ans.} \end{array}$$

(*b*)
$$\begin{array}{r} 493 \\ \times .125 \\ \hline 61.625 \textit{ Ans.} \end{array}$$

In examples indicating multiplication by a decimal, the pupils should employ a common fraction as the multiplier when the work is shortened by the change, even when the multiplier is written as a decimal. Thus the products by .25 and .125, respectively, are obtained by using as multipliers $\frac{1}{4}$ and $\frac{1}{8}$, respectively. A similar practice may be employed when the indicated multiplier is a fraction, the equivalent decimal being actually used. In finding $\frac{4}{5}$ of 493, the product by .8 is obtained in a single line; in writing the result $\frac{2}{5}$ is substituted for .4, the right-hand figure of 3 × .8. In like manner .08 is substituted for the factor $\frac{2}{25}$, the decimal .44 being reduced to $\frac{11}{25}$ in writing the product.

(*c*) $\frac{4}{5}$ of $493 = 394\frac{2}{5}$

(*d*) $\frac{2}{25}$ of $493 = 39\ \frac{11}{25}$

It is by no means recommended that all possible methods of shortening work should be suggested, as this would entail a serious loss of time without corresponding benefit. Only such are advised as appeal to the intelligence of the average pupil. Most of the short methods should be brought out by the pupils' answers to questions as to a possible abbreviation of long-drawn-out work of a pupil at the blackboard.

(*a*)
$$\begin{array}{rrl} & 5\frac{1}{2} & \\ & \underline{5\frac{1}{2}} & \\ & 25 & (5 \text{ times } 5) \\ (1 \text{ times } 5) \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{r} 2\frac{1}{2} \\ 2\frac{1}{2} \end{array} & \begin{array}{l} (\frac{1}{2} \text{ `` } 5) \\ (\frac{1}{2} \text{ `` } 5) \end{array} \\ & \frac{1}{4} & (\frac{1}{2} \text{ `` } \frac{1}{2}) \\ \cline{2-2} \textit{Ans.} & 30\frac{1}{4} & (6 \text{ times } 5 + \frac{1}{2} \text{ of } \frac{1}{2}) \end{array}$$

For the information of the teacher, some combinations are here presented to be used at his discretion. One type is illustrated by the

$$\begin{array}{r@{\quad}r@{\quad}l}
(b) & 8\frac{1}{3} & \\
 & 8\frac{2}{3} & \\ \cline{2-2}
 & 64 & (8 \text{ times } 8) \\
(1 \text{ times } 8) \left\{ \begin{array}{c} \\ \\ \end{array} \right. & \begin{array}{r} 5\frac{1}{3} \\ 2\frac{2}{3} \end{array} & \begin{array}{l} (\frac{2}{3} \quad \text{``} \quad 8) \\ (\frac{1}{3} \quad \text{``} \quad 8) \end{array} \\
 & \frac{2}{9} & (\frac{1}{3} \quad \text{``} \quad \frac{2}{3}) \\ \cline{2-2}
 & 72\frac{2}{9} & (8 \text{ times } 9 + \frac{1}{3} \text{ of } \frac{2}{3})
\end{array}$$

cross-multiplication of (*a*) $5\frac{1}{2}$ by $5\frac{1}{2}$ and that of (*b*) $8\frac{1}{3}$ by $8\frac{2}{3}$. In (*a*) 5 is multiplied. by 5, then 5 in the multiplicand is multiplied by $\frac{1}{2}$ in the multiplier, next 5 in the multiplier is multiplied by $\frac{1}{2}$ in the multiplicand, the final product being that of the two fractions. It will be noticed that the integers in the two factors being the same and the sum of the two fractions being 1, the sum of the second and third partial products is equal to the given whole number. In finding, therefore, the product of two mixed numbers when both integers are the same and the sum of the fractions is 1, one of the integers is increased by 1, is then multiplied by the other, and to their product is affixed the product of the fractions; as, for example:

(*c*) $3\frac{3}{4} \times 3\frac{1}{4} = (4 \times 3) + (\frac{1}{4} \text{ of } \frac{3}{4}) = 12\frac{3}{16}$
(*d*) $9\frac{1}{5} \times 9\frac{4}{5} = (9 \times 10) + (\frac{1}{5} \text{ of } \frac{4}{5}) = 90\frac{4}{25}$

The same method is followed when the common fractions are replaced by decimals.

(*e*) $4.3 \times 4.7 = (5 \times 4) + (.3 \times .7) = 20.21$
(*f*) $6.1 \times 6.9 = (7 \times 6) + (.1 \times .9) = 42.09$

When the factors are integers having the tens' figures alike and the sum of the ones' figures equal to 10, the results are those obtained in (*e*) and (*f*), with the omission of the decimal point.

(*g*) $43 \times 47 = (5 \text{ tens} \times 4 \text{ tens}) + (3 \times 7) = 2021$
(*h*) $61 \times 69 = (7 \text{ tens} \times 6 \text{ tens}) + (1 \times 9) = 4209$
(*i*) $22 \times 28 = (3 \text{ tens} \times 2 \text{ tens}) + (8 \times 2) = 616$
(*j*) $74 \times 76 = (8 \text{ tens} \times 7 \text{ tens}) + (4 \times 6) = 5624$

In practice the pupil does not think in (*j*), for instance, that the product of 8 tens by 7 tens gives 56 hundreds; he merely uses 8 by 7 to obtain the first two figures of the total, and 4 by 6 to obtain the last two.

54321 × 11
54321
597531 *Ans.*

Some state courses of study include this last type of combinations in the prescribed oral drill exercises, and among the products required to be written in one line are those by multiples of 11. As a preliminary to the method for finding multiples of 11, the two partial products of 54321 multiplied by 11 are given, with their sum. Beginning at the right, it will be observed that the first figure of the product is the first figure of the multiplicand, that the second figure of the product is the sum of the first and second figures of the multiplicand, etc., as is shown above. In multiplying 54321 by twice 11, or 22, the first figure of the product is twice the first figure of the multiplicand, the second figure of the product is twice the sum of the first and second figures of the multiplicand, the third figure of the product is the ones' figure of twice the sum of the second and third figures of the multiplicand, the tens' figure being carried to the next, etc.

First	figure	of	product	1
Second	"	"	"	1 + 2
Third	"	"	"	2 + 3
Fourth	"	"	"	3 + 4
Fifth	"	"	"	4 + 5
Sixth	"	"	"	5

1 × 2	gives	1st	figure.	Write	2.		
(1 + 2) × 2	"	2d	" .	"	6.		
(2 + 3) × 2	"	3d	" .	"	0.	Carry 1	
[(3 + 4) × 2] + 1	"	4th	" .	"	5.	" 1	
[(4 + 5) × 2] + 1	"	5th	" .	"	9.	" 1	
(5 × 2) + 1	"	6th and 7th.		"	11.		*Ans.* 1,195,062

The product of 65432 by 88 is written in one line as follows:

8	times	2	= 16.				Carry	1,	write	6
8	"	(2 + 3)	= 40;	adding	1 gives	41.	"	4,	"	1
8	"	(3 + 4)	= 56;	"	4 "	60.	"	6,	"	0
8	"	(4 + 5)	= 72;	"	6 "	78.	"	7	"	8
8	"	(5 + 6)	= 88;	"	7 "	95.	"	9	"	5
8	"	6	= 48;	"	9 "	57.			"	57

65432 × 88 = 5,758,016 *Ans.*

Pupils who have used in the lower grades in their oral and sight exercises such multipliers as $12\frac{1}{2}$, $16\frac{2}{3}$, $33\frac{1}{3}$, $37\frac{1}{2}$, $62\frac{1}{2}$, $66\frac{2}{3}$, $87\frac{1}{2}$, etc., are frequently interested in easy combinations that employ as one factor a number larger or smaller than the foregoing by 1, by $\frac{1}{2}$, by $\frac{1}{4}$, by $\frac{1}{8}$, etc., which are similar to some of the preceding fractional combinations.

After a pupil has found the cost of 96 articles at $12\frac{1}{2}$ cents each, he should be given examples in which the respective prices are $\frac{1}{8}$¢, $\frac{1}{4}$¢, $\frac{1}{2}$¢, 1¢ more than $12\frac{1}{2}$¢. Thus 96 articles at $12\frac{5}{8}$ cents each will cost \$12 + $\frac{1}{8}$ of 96 cents, or \$12.12; at $12\frac{3}{4}$¢, \$12 + $\frac{1}{4}$ of 96 cents, or \$12.24; at 13¢, \$12 + $\frac{1}{2}$ of 96 cents, or \$12.48; at $13\frac{1}{2}$¢, \$12 + 96 cents, or \$12.96. At prices $\frac{1}{8}$¢, $\frac{1}{4}$¢, $\frac{1}{2}$¢, 1¢, respectively, less than $12\frac{1}{2}$¢, deductions are made as follows: 96 articles at $12\frac{3}{8}$¢ will cost \$12 − $\frac{1}{8}$ of 96 cents, or \$12 − 12¢, or \$11.88; at $12\frac{1}{4}$ cents, \$12 − 24¢, or \$11.76; at 12 cents, \$12 − 48¢, or \$11.52; at $11\frac{1}{2}$¢, \$12 − 96¢, or \$11.04.

Many teachers employ combinations of this kind in the bills dictated to pupils, the latter being required to write the extensions without "side" calculations. The following are some items:

48 pounds of coffee, @ $12\frac{3}{4}$¢
36 pounds of cocoa, @ 34¢
66 yards of silk, @ \$1.35
$24\frac{3}{4}$ pounds of butter, @ 32¢
24,750 feet of boards, @ \$48 per M.
96 bushels of wheat, @ $99\frac{7}{8}$¢

Quantities and prices found in the text-books may be so altered as to supply other sets of factors.

Some of the foregoing combinations may be used in sight exercises in finding areas of rectangles, and may also be presented in the decimal form:

$$\begin{array}{llllllll}
(a) & \begin{array}{r} 88 \\ \times\ 12.5 \\ \hline \end{array} & (b) & \begin{array}{r} 88 \\ \times\ 12.625 \\ \hline \end{array} & (c) & \begin{array}{r} 12.75 \\ \times\ 88 \\ \hline \end{array} & (d) & \begin{array}{r} 13.5 \\ \times\ 88 \\ \hline \end{array} \\
(e) & \begin{array}{r} 64 \\ \times\ 11.5 \\ \hline \end{array} & (f) & \begin{array}{r} 12.375 \\ \times\ 48 \\ \hline \end{array} & (g) & \begin{array}{r} 56 \\ \times\ 12.25 \\ \hline \end{array} & (h) & \begin{array}{r} 1.3 \\ \times\ 72 \\ \hline \end{array}
\end{array}$$

The process of finding the square of a number composed of tens and ones is frequently taught by the algebraic method of squaring the sum of two numbers:

$$\begin{aligned} 37^2 = (30 + 7)^2 &= 30^2 + 30 \times 2 \times 7 + 7^2 \\ &= 900 + 420 + 49 \end{aligned}$$

When the number ends in 9, it may be considered as the difference of two numbers; thus:

$$39^2 = (40 - 1)^2 = 40^2 - 40 \times 2 \times 9 + 1^2$$

The product of two such numbers as 52 and 48, 37 and 43, is obtained by the formula for the product of the sum of two numbers by their difference:

$$52 \times 48 = (50 + 2) \times (50 - 2) = 50^2 - 2^2 = \\ 2500 - 4 = 2496$$

$$37 \times 43 = (40 - 3) \times (40 + 3) = 40^2 - 3^2 = \\ 1600 - 9 = 1591$$

An occasional example should be placed on the blackboard involving, for instance, (*a*) the sum of 846 and 7 times 2357, or (*b*) the deduction of 9 times 427 from 4000, the pupils writing only the final result. In (*a*) the pupil says:

7 times 7 (49); add 6, 55. Write 5
7 times 5 (35); 5 carried (40); add 4, 44. Write 4
7 times 3 (21); 4 carried (25); add 8, 33. Write 3
7 times 2 (14); 3 carried, 17. Write 17
Ans. 17345.

The same operations are involved in the reduction of $7\frac{846}{2357}$ to an improper fraction, all examples of this kind being worked by "adding in" each figure of the numerator to the corresponding figure of the product.

In (*b*), 4000 − (9 × 427), the pupil says:

9 times 7 (63) and 7 are 70. Write 7
9 times 2 (18), carry 7 (25), and 5 are 30. Write 5
9 times 4 (36), carry 3 (39), and 1 are 40. Write 1
Ans. 157.

The same operations are involved in the reduction of $\frac{4000}{427}$ to the mixed number, $9\frac{157}{427}$.

When an improper fraction is reducible to a mixed number the integral part of which is less than 13, the result may be obtained in this way.

If it be thought desirable to teach the method of testing the accuracy of a product by "casting out the nines," a few preparatory exercises may be given. The pupils already know that a number is divisible by 9 when the sum of its digits is divisible by 9; and the fact may be developed that when the division of this sum by 9 is inexact, the remainder is the same as the remainder of the number itself divided by 9. Thus 14863 ÷ 9 = 1651 quotient, 4 remainder; $(1 + 4 + 8 + 6 + 3) \div 9 = 22 \div 9 = 2$ quotient, 4 remainder. This means that the excess of nines in a given number is obtained by finding the excess in the sum of the digits. Without going into the formal proof, pupils are willing to believe that the product of the excess of nines in one factor by the excess in another factor should have an excess equal to the excess found in the product of the factors themselves, especially when they find this to be true after testing several examples.

The method is shown in (*a*), in which 14863 is multiplied by 2049.

When 14863 is divided by 9 there is a remainder of 4, which is also obtained by adding the digits and dividing their sum by 9. In practice each nine is rejected as it is reached, 3 + 6 = 9, rejected; 8 + 4 = 12, excess 3, which is added to 1, making 4 the excess for the multiplicand. In finding the excess of the multiplier, 9 is rejected, and the sum of 4 and 2 gives the excess 6, which is the remainder when 2049 is divided by 9. The product of the excesses (6 × 4) gives 24, the excess of which is 4 + 2, or 6. The excess of the product is also 6, which must be the case when the multiplication is correctly performed.

(*a*)	14863	sum of digits 22; excess	4
	× 2049	" " " 15; "	6
	133767	4 × 6	24
	59452	excess	6
	29726		
Ans.	30454287	sum of digits 33; excess	6

In (*b*) an error made in locating the third partial product gives a wrong result, which is not detected in casting out nines; and the error in (*c*) made by taking 9 eights as 63 in the first partial product also escapes detection by this form of proof.

Incorrect
(*b*) 14863
×2049
133767
59452
29726
3700887
sum of digits 33
excess 6

Incorrect
(*c*) 14863
× 2049
132867
59452
29826
30453387
sum of digits 33
excess 6

In order, therefore, to make certain that a product is correct when it meets the test of "casting out nines," a second test is also applied, "casting out elevens."

A number is divisible by 11 when the sum of the alternate digits beginning with the ones is equal to the sum of the alternate digits beginning with the tens, or when the excesses of these sums are equal. This is shown in the following products, each of which is, of course, divisible by 11:

$$\begin{array}{rrrrrrrr} & 145 & & 237 & & 732 & & 158 \\ & \times\ 11 & & \times\ 11 & & \times\ 11 & & \times\ 11 \\ \cline{2-2}\cline{4-4}\cline{6-6}\cline{8-8} (d) & 1595 & (e) & 2607 & (f) & 8052 & (g) & 1738 \end{array}$$

In (*d*) $5 + 5 = 9 + 1$
In (*e*) $7 + 6 = 13$; excess $2: 0 + 2 = 2; 2 = 2$
In (*f*) $2 + 0 = 2; 5 + 8 = 13$; excess $2: 2 = 2$
In (*g*) $8 + 7 = 15$; excess $4: 3 + 1 = 4; 4 = 4$

The excess of elevens in a product is obtained by deducting the sum of the alternate digits beginning with the tens, from the sum of the alternate digits beginning with the ones. When the former sum is greater than the latter, as in (*f*), its excess may be used, or the latter may be increased by 11 when necessary.

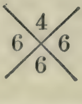
The excess of elevens in the multiplicand of (*a*) is obtained by adding the alternate digits beginning with the ones, $3 + 8 + 1 = 12$, and deducting from this sum the total of the remaining digits, $6 + 4 = 10$, giving 2 as the excess. The excess in the multiplier is $(9 + 0) - (4 + 2)$, or 3. The product of these excesses is (3×2) 6. The excess in the product is $(7 + 2 + 5 + 0) - (8 + 4 + 4 + 3)$, or $14 - 8$ (the excess), which is 6. This latter result being the same as the product of the excesses in the two factors, the answer is presumably correct. It is certainly so when the test by casting out the nines shows a like condition.

The test of the product in (*b*) by casting out elevens gives the following:

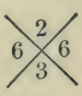
$$(7 + 8 + 0 + 3) - (8 + 0 + 7) = 3$$

As this differs from 6, the product of the excesses of the factors, the result is incorrect, notwithstanding the fact that it stands the other test. The test of the product in (*c*) by casting out elevens gives the following result: $(7 + 3 + 5 + 0) - (8 + 3 + 4 + 3)$ or $15 - 18$, or $15 - 7$ (the excess), which is 8. This also detects the inaccuracy of the work notwithstanding the result of the other test.

In making the tests the excesses are sometimes arranged in the four spaces of a multiplication sign, the excess obtained by casting out nines in the multiplicand of example (*a*), for instance, being written in the upper space and the excess in the multiplier in the lower. The excess of the product of these excesses is written on the right. The excess of the product of the given numbers, which is written on the left, should be the same.



The test by casting out elevens gives results shown in this form, 2 being the excess in the multiplicand, 3 the excess in the multiplier. The right-hand 6 is the product of 3 by 2, and agrees with the 6 on the left, which is the excess in the product.



Even when the pupils are not to be taught to apply these tests, the teacher should familiarize himself with them, and employ them occasionally in checking products and quotients arising in his own work. A teacher of elementary arithmetic should be well acquainted with high-school mathematics to be entirely successful in the lower grades.

Contracted multiplication is seldom employed below the high school, but an example is shown worked in two ways, together with the same example in which all the figures are retained.

Find the product of 3.141593 by 21.526475 correct to two decimal places.

```
(a) |3.1415|93
 574|625.12|
     62.832
      3.142
      1.571
       .063
       .019
     67.627
   67.63 Ans.
```

The method (*a*), followed by many accountants, reverses the digits of the multiplier, placing the ones' figure under the third decimal place of the multiplier, or one place more than is called for in the result. Beginning at the left, 3.1415 is multiplied by 2 tens, and the product written, three decimal places being pointed off. When the partial product by 1 is found, the work is begun with the figure above 1, the multiplicand now being 3.141. The next multiplier is 5, and the multiplicand 3.14; the last figure of each successive multiplicand being the one immediately above the multiplier then employed. As there are no figures in the multiplicand above 574, these figures are not used as multipliers. In each case, however, the effect of the figure to the right is considered, and the proper number is carried.

```
(b)          3.141593
            21.526475
             15707965
            21991151
          1|2566372
         18|849558
         62|83186
       1570|7965
       3141|593
      62831|86
     67.627|423174675
         67.63 Ans.
```

In (*b*) all the figures are used, the result being the same as far as the first three decimal places, and the answer is abbreviated by increasing the hundredths' figure by 1, since the thousandths' figure, 7, is greater than 5. In (*c*) the decimal point in the multiplier is moved one place to the right, leaving only one digit to the left of the decimal point. Since the multiplier has thus been divided by 10, the multiplicand must be multiplied by 10, which is done by moving the decimal point one place to the left, making it 31.41593. As two places are needed in the result, an additional place is retained in the multiplicand, the last figure, 5, being changed to 6, since the succeeding figure is 9. There being five digits in the multiplicand, the same number is retained in the multiplier. The work begins with 2 as the multiplier, the partial product having three decimal places. The last figure of the multiplicand is then canceled, and .1 is taken as the second multiplier. The effect of the canceled 6 on the product is considered, however, 1 being carried to the first figure written. The 1 of the multiplicand is then canceled, and the remaining figures are multiplied by .05, 1 being carried, notwithstanding the cancellation. The 4 of the multiplicand is next canceled, and the product by .002 written. Finally the next 1 of the multiplier is canceled and the product of 3 tens by .0006 is written, 1 being again carried.

$$\begin{array}{r} (c)\ \cancel{3}\cancel{1}.\cancel{4}\cancel{1}\cancel{6} \\ \underline{2.1526} \\ 62.832 \\ 3.142 \\ 1.571 \\ .063 \\ \underline{.019} \\ 67.627 \\ 67.63\ \mathit{Ans.} \end{array}$$

While this method retains the regular order of the digits in the multiplier, the decimal point in the multiplier is placed after the first figure and the necessary change made in the multiplicand. The popularity of the first method is due to the arrangement that begins each successive multiplication with the figure immediately above the multiplier.

In multiplying two decimals, the pupil should consider the probable result. Business men point off the integral portion rather than the decimal one. In the product of 3.1416 by 21.526, they place the period after 67 of the result 67.627 etc., without thinking about the rule. They know that the result is something more than 63 and they locate the decimal point accordingly.

The relation between multiplication and division may be shown by an example such as the cost of 247 barrels of flour

$$\begin{array}{r}(a)\ \$5\\ \times\ 247\\ \hline \$1235\end{array}$$

at $5 per barrel; in this the cost of 1 barrel is to be multiplied by the *number* of barrels, although in practice the smaller factor is used as the multiplier. The pupils should be able to state the two division problems that are derived from the foregoing, viz.:

(*b*) "Find the cost per barrel when 247 barrels can be bought for $1235."

(*c*) "Find the number of barrels at $5 each that can be bought for $1235."

The first, (*b*), is stated in this way:

$$\$1235 \div 247 = \$5, \text{ the cost per barrel.}$$

The second, (*c*), is stated thus:

$$\$1235 \div \$5 = 247, \text{ the } \textit{number} \text{ of barrels.}$$

The answer to (*c*) is written 247 bbl., the denomination being supplied. The pupils may be called upon to note that the denomination is omitted after 247 in the multiplication example (*a*), this number being the *ratio* between 247 barrels and 1 barrel, the whole operation being $\$5 \times \frac{247\text{ bbl.}}{1\text{ bbl.}}$ This may be made clear by the following:

(*d*) "Find the rate per dozen for oranges when they are sold at 3 for 5 cents."

$$\begin{array}{r}5\,¢\\ \times\ 4\\ \hline 20\,¢\end{array}$$

In this example the cost of 3 oranges is multiplied by 4, the ratio between 12 oranges and 3 oranges, which is an abbreviated form of

$$5¢ \times \frac{12 \text{ oranges}}{3 \text{ oranges}}.$$

The teacher must determine for himself how far it is advisable to go into the question of ratio in these illustrations, the main point to be developed being that a dividend may be either abstract or concrete, the divisor in the former case being necessarily abstract (*e*), and in the latter case either abstract (*f*), or concrete (*g*).

$$(e)\ \begin{array}{r}5)\underline{85}\\17\end{array}\qquad (f)\ \begin{array}{r}\$5)\underline{\$85}\\17\end{array}\qquad (g)\ \begin{array}{r}5)\underline{\$85}\\\$17\end{array}$$

The chief use of the discussion is to bring out the fact that in (*f*) \$5 is contained 17 times in \$85, and to show the connection between this and the fact that 5 sevenths is contained 17 times in 85 sevenths, 5 hundredths is contained 17 times in 85 hundredths, and the like.

English mathematicians do not share our notions as to abstract multipliers. The following are typical instances:

"1. *Under what conditions can two concrete quantities of the same kind be multiplied together?*

"A length may be multiplied by a length if both be expressed in the same denomination, and the result will be an area."

"2. *Under what conditions is it possible to multiply a concrete number of one kind by a concrete number of another kind?*

"An area expressed in square units of any kind may be multiplied by a length expressed in units of the same kind, to give a volume expressed in solid units of the same kind."

"3. *Give three cases in which a concrete quantity of one kind can be divided by a concrete quantity of another kind.*"

The illustrations given are (I) the area of a rectangle 145 sq. ft. divided by the breadth 5 ft.; (II) the volume of a prism 1700 cu. ft. divided by its height 17 ft.; and (III) the volume 1700 cu. ft. divided by the area of the base 100 sq. ft.

The author of the foregoing questions and answers is a lecturer in mathematics at King's College, University of London. They are presented to warn teachers that elementary text-books written down to the level of immature minds contain statements necessarily brief, that do not express the whole truth, or that present only one side of a disputed point. While a pupil should be taught, perhaps, that the area of a rectangle is obtained by multiplying the number of square units in one row by the number of rows, it is not necessary to teach what is not true, that feet are never multiplied by feet.

Before working an example in long division on the blackboard, a pupil should state the number of quotient figures. The method of dividing without writing the partial products may be shown by means of a few examples worked by the pupils. Although abbreviated division is considered difficult by teachers in this country, many millions of children have never used any other form. The quotient of 14739 by 187 is obtained as follows:

$$\begin{array}{r} \underline{\textit{Ans.}\ 78\tfrac{153}{187}} \\ 187)14739 \\ 1649 \\ 153 \end{array}$$

Multiplying 187 by the first quotient figure, 7, the remainder is written directly, the pupil saying (or thinking):

7 times 7 (49) and 4 are 53. Write 4
7 times 8 (56), carry 5 (61), and 6 are 67. Write 6
7 times 1 (7), carry 6 (13), and 1 are 14. Write 1

To the remainder, 164, 9 is annexed, the divisor is multiplied by the second quotient figure, 8, and the remainder is written:

8 times 7 (56) and 3 are 59. Write 3
8 times 8 (64) carry 5 (69), and 5 are 74. Write 5
8 times 1 (8) carry 7 (15), and 1 are 16. Write 1

Each figure of the remainders is written when it is spoken after the word "and," although in the foregoing illustration, the instruction to write it is given last. After the first figure, 4, of the first remainder is written, the pupil says 53, which indicates that 5 is to be carried to the next product, etc.

The process is further illustrated by some examples in reduction of improper fractions, as was shown on a previous page, the following being typical:

(*a*) $\frac{475}{84} = 5\frac{?}{84}$ (*b*) $\frac{1000}{161} = 6\frac{?}{161}$ (*c*) $\frac{1100}{89} = 12\frac{?}{89}$

A second set requires the pupils to determine the whole number in each case:

(*d*) $\frac{439}{74}$ (*e*) $\frac{523}{106}$ (*f*) $\frac{875}{92}$ (*g*) $\frac{1234}{106}$

$$\begin{array}{rl} (h) & 106 \div \frac{3}{4} \\ \text{Add } \frac{1}{3} & \underline{35\frac{1}{3}} \\ & 141\frac{1}{3}\ \textit{Ans.} \end{array}$$

Pupils should be given some acquaintance with the business man's method of dividing by $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{7}{8}$, etc. Instead of using the inverted fractions in their improper forms, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{6}{5}$, $\frac{8}{7}$, etc., the accountant employs the mixed numbers, $1\frac{1}{3}$, $1\frac{1}{4}$, $1\frac{1}{5}$, $1\frac{1}{7}$, etc., by adding to the multiplicand $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{7}$, etc., respectively, of itself, especially when the multiplicand is an integer. The same method is followed when one of the two factors of a compound numerator exceeds the denominator by 1, as in (*i*).

$$(i)\ \frac{106 \times 4}{3} \qquad 141\tfrac{1}{3}\ \textit{Ans.}$$

The review drills should include all of the types previously studied, together with a few new ones. The following are examples:

(*a*) 1000 ÷ 99 (*b*) 2200 ÷ 99 (*c*) 2213 ÷ 99
(*d*) 1000 ÷ 98 (*e*) 2200 ÷ 98 (*f*) 2200 ÷ 97
(*g*) $200\frac{25}{29} \div 50$ (*h*) $140\frac{20}{21} \div 70$ (*i*) $780\frac{12}{13} \div 60$
(*j*) 189 ÷ .7 (*k*) 325 ÷ .05 (*l*) 141 ÷ .003
(*m*) 288 ÷ 3.2 (*n*) 30.25 ÷ 5.5 (*o*) 42.25 ÷ 6.5
(*p*) 99 ÷ 1.125 (*q*) 99 ÷ 1.375 (*r*) 12 ÷ .1875
(*s*) 11 ÷ .0625 (*t*) 4.4 ÷ .6875 (*u*) 4.5 ÷ .9375

The pupils are frequently interested to know that the answer to such an example as (*v*) 123423 ÷ 99 can be written at sight. When they know that 1200 ÷ 99 gives a quotient of 12 with a remainder of 12, they can understand that 1234 ÷ 99 gives a quotient of 12 with a remainder of 12 + 34. This may be extended to (*w*) 123400 ÷ 99, the first quotient of which may be considered as 1234 with a remainder of 1234, which is yet to be divided. The second quotient is 12 with a remainder of 12 + 34, the entire quotient being 1234 + 12, and the remainder 12 + 34. When the dividend is further increased by 23, the quotient is 1234 + 12 (1246), and the remainder is 12 + 34 + 23 (69). The quotient is, therefore, 1246; remainder, 69. The successive steps are shown as follows:

$$\begin{array}{rrl}
(v)\ 123423 \div 99 = & 1234; & \text{rem. } 23\ (+\ 1234) \\
1234 \div 99 = & 12; & \text{rem. } 34\ (+\ 12) \\
12 \div 99 = & & \text{rem. } 12 \\
\cline{2-3}
 & 1246; & \text{rem. } 69
\end{array}$$

The result is written at sight by combining 1234, the first four figures, with 12, the first two, for the quotient; and the three pairs 12 + 34 + 23 for the remainder. In dividing 123456 by 99, the sum of the three pairs is 102, which is 3 more than 99. In this case, 1246 is increased by 1, making the quotient 1247, the remainder being 3.

$$\begin{array}{r}
1234;\ 56 \\
12;\ 34 \\
12 \\
\hline
1246;\ 102
\end{array}$$

quotient 1247;
rem. 3 *Ans.*

The quotient of 1234321 divided by 99 is obtained by writing the last two figures apart from the others as a first remainder, the figures that are left being the first quotient. The second remainder consists of the last two figures of the first quotient, the remaining figures constituting the second quotient. The third remainder consists of the last two figures of the

$$\begin{array}{r}
12343;\ 21 \\
123;\ 43 \\
1;\ 23 \\
1 \\
\hline
12467;\ 88
\end{array}$$

second quotient, the remaining figure constituting the third quotient and also the fourth remainder. When the sum of the remainders exceeds 99, it must be reduced; as in the following in which 123456789 is divided by 99. The successive partial quotients and partial remainders are written as before. The sum of the remainders being 225, this is divided by 99, giving 2 for quotient and 27 for remainder. The latter is written and 2 is carried to the other quotients. The division of 123456 by 999 gives a quotient of 123 and a remainder of 123 + 456.

$$\begin{array}{r}
1234567;\ 89 \\
12345;\ 67 \\
123;\ 45 \\
1;\ 23 \\
1 \\
\hline
1247036;\ 225 \\
\text{quotient } 1247038;\ \text{rem. } 27\ \textit{Ans.}
\end{array}$$

In finding the quotient of (*n*) the pupil is supposed to recall the product 5.5×5.5, or $5\frac{1}{2} \times 5\frac{1}{2}$. The same is true of (*o*). In (*p*), $99 \div 1.125$, the decimal is changed to a fraction, the example becoming $99 \div 1\frac{1}{8}$, or $99 \div \frac{9}{8}$, or $99 \times \frac{8}{9}$, in which shape it is reasonably simple. In (*r*) the decimal .1875 is changed to the fraction $\frac{3}{16}$; in (*t*) .6875 is changed to $\frac{11}{16}$; in (*u*) .9375 is changed to $\frac{15}{16}$.

Much less time should be devoted to division exercises than to those in multiplication.

$$\begin{array}{r}
5461 \\
2197\overline{)12000000} \\
10150 \\
13620 \\
4380 \\
2183 \\
\text{quotient } 5461 \\
\text{rem. } 2183
\end{array}$$

The test of the correctness of a division result by casting out nines and elevens is similar in principle to that employed in multiplication. In the accompanying example, the dividend 12,000,000 is equal to the remainder, 2183, added to the product of the divisor, 2197, by the quotient, 5461.

Casting out nines:

The excess in 2197, the divisor, $= 1$
The excess in 5461, the quotient, $= \underline{7}$

The excess in their product is 7 × 1 = 7
Adding the excess in the remainder, 2183 = $\underline{5}$

The sum is 12, of which the excess = 3
This corresponds with the excess in the dividend, 1 + 2 = $\underline{3}$

Casting out elevens:

The excess in the divisor is (7 + 1) − 0 [the excess of (9 + 2)] = 8
The excess in the quotient is (1 + 4) − 0 [the excess of (6 + 5)] = $\underline{5}$

The excess in their product 5×8, or 40 = 7
Adding the excess in the remainder, which is [(3 + 1) increased by 11] − (8 + 2) = $\underline{5}$

The sum is 12, of which the excess = 1
This corresponds with the excess in the dividend, which is 2 − 1 = $\underline{1}$

CHAPTER II

SEVENTH-YEAR PERCENTAGE

When percentage is not taught in the sixth year, the work in the seventh grade may begin in the manner suggested in a previous chapter. In his treatment of the subject at any stage, the teacher should be careful to avoid leading the pupils to imagine they are learning anything new other than the meaning of certain commercial terms that have not found place in the arithmetic exercises of the lower grades.

After the pupils have completed the work previously outlined, they may read and discuss in class the meaning of the various terms given under the heading of "Applications of Percentage." The discussion should bring out the fact there is nothing mathematically new in the new words, and that there is no necessity for a separate treatment of commission, insurance, taxes, etc., other than the desire to avoid the introduction of too many strange terms at one time. In order to give the pupils a greater interest in these applications, an insurance policy, a tax-bill, and the like may be brought to the class-room; but the teacher should be careful to devote only a minimum of time to these separate topics, and to teach only such as are prescribed. The pupils will not develop in mathematics by being told in several different ways to multiply the base by the rate. To complete this portion of the work in the shortest time possible, the problems should be read aloud in class, each pupil stating his method of obtaining the result, and the approximate answer.

Before taking up the indirect cases, the teacher should give many examples, oral and written, employing rates less than one per cent and others greater than 100%, which may be preceded by such preliminary exercises in reductions as the following:

1. Express as common fractions:

(*a*) 4% (*b*) 25% (*c*) $\frac{1}{2}$% (*d*) $\frac{1}{4}$% (*e*) $112\frac{1}{2}$%

2. Express as decimals:

(*a*) $\frac{1}{2}$% (*b*) $\frac{1}{4}$% (*c*) $12\frac{1}{2}$% (*d*) $\frac{1}{8}$% (*e*) 150%

3. Express as per cents:

(*a*) $\frac{1}{200}$ (*b*) .375 (*c*) $\frac{3}{200}$ (*d*) .0625 (*e*) $1\frac{5}{8}$

The text-books supply many other examples, all of which should be used until the types are fully mastered.

The oral and written examples mentioned above should contain such rates as $\frac{1}{8}$%, $\frac{1}{4}$%, $\frac{3}{4}$%, 150%, 1275%, in order to familiarize the pupils with the difference between $\frac{1}{8}$% and $12\frac{1}{2}$%, $\frac{3}{4}$% and 75%, etc., and also to show that per cents greater than 100% are not uncommon.

To emphasize the fact that $\frac{1}{2}$%, $\frac{3}{4}$%, $\frac{5}{8}$%, etc., are less than 1%, business men state them as $\frac{1}{2}$ of 1 per cent, $\frac{3}{4}$ of 1 per cent, etc.

The two chief indirect cases should be taken up together, and in connection with fraction and decimal exercises suggested in the chapter devoted to Relation of Numbers in Part Three.

Such an example as the relation of 12 to 15 may be presented in the following ways:

1. (*a*) "What fraction of 15 is 12? (*b*) What decimal? (*c*) What per cent?"

2. (*a*) "15 is what fraction of 12? (*b*) What decimal? (*c*) What per cent?

The other case may be presented in the following ways:

3. "(*a*) When $\frac{3}{8}$ of a number is 24, what is the number? (*b*) When .375 of a number is 36, what is the number? (*c*) When $37\frac{1}{2}\%$ of a number is 18, what is the number?"

4. "Find the number (*a*) when $\frac{9}{8}$ of it is 27. (*b*) When 1.125 of it is 36. (*c*) When $112\frac{1}{2}\%$ of it is 45."

Many a pupil who has no difficulty in getting the answers to the foregoing as oral exercises, blunders when he works the written ones. In finding what fraction of 15 is 12, he divides 12 by 15, obtaining the result in the fractional form as $\frac{4}{5}$, or in the decimal one as .8, either of which he reduces to 80 hundredths, which he then translates into 80 per cent. In doing this, he really multiplies $\frac{4}{5}$ or .8 by 100, and then affixes the per cent sign, %. The work is done, therefore, in two steps, the first being $12 \div 15$, and the second being the product of this quotient by 100%, which may be represented thus:

$$\left(\frac{12}{15}\right) \times 100\%$$

Some successful teachers encourage the two-step method of first obtaining the answer as a fraction or a decimal and multiplying the result by 100%. Others, preferring to indicate at once all the operations necessary to obtain 80%, advise the writing of the foregoing in this form:

$$\frac{12 \times 100\%}{15}$$

While the pupils by this method work a number of examples correctly when the latter are brought together and are classified as Case 2 (or 3), they fail to obtain a complete grasp of the underlying reasons. They do not understand that in multiplying $\frac{12}{15}$ by 100 and annexing the per cent sign they are multiplying 100% by $\frac{12}{15}$, and that 100% is $\frac{100}{100}$, or 1.

The formula method, which derives $R = \frac{P \times 100}{B}$ from $P = \frac{B \times R}{100}$, is unsatisfactory in practice. Pupils who have occasion to use formulas in later work should learn that it is unnecessary to burden their memories with the derived ones. The substitution of the given numbers in the original formula, and the solution of the resulting equation, is much more profitable.

The boy or the girl who multiplies the base by the rate in hundredths to obtain the percentage will readily write in the form of an equation the following:

"What per cent of 15 is 12?"

If the teacher has no unnecessary scruples concerning the employment of x, the pupil sets down at once:

$$\frac{x}{100} \times 15 = 12$$

writing x for "what," 100 as a denominator for "per cent," × for "of," and the sign of equality for "is." If there be an objection to the harmless x, it can be replaced by R, thus:

$$\frac{R}{100} \times 15 = 12$$

When the problem reads:

"12 is what per cent of 15?"

the equation may first be written in the following form, which takes up the numbers and signs in the order in which they occur in the problem:

$$12 = \frac{R}{100} \times 15$$

which is then changed to the form given above.

The particular advantage of this method is that it prevents the pupil from employing the wrong number as the divisor. In the problems:

(*a*) "15 is what per cent of 12?"

(*b*) "What per cent of 12 is 15?"

the thoughtless boy or girl not infrequently selects the larger number as the base, while by the equation method there is no need to give any attention to the matter,

$$15 = \frac{R}{100} \times 12 \quad \text{being written for } (a)\text{, and}$$

$$\frac{R}{100} \times 12 = 15 \quad \text{being written for } (b).$$

Experienced people sometimes have difficulty in determining which number is the divisor, and not a few teachers in their own work employ the old-time rule, "Divide the 'is' number by the 'of' number," even when they would not consider it right to have a pupil hear of its existence.

The equation method is particularly helpful in the case of very small or very large per cents. The rates in the average example are generally between 1 and 99, and the careless pupil is not unlikely to give 25% as the result when the correct one is either $\frac{1}{4}$% or 2500%. Although this is less apt to occur when answers are tested, it is not uncommon. In the two following examples:

(*a*) "What per cent of 900 is $2\frac{1}{4}$?"

(*b*) "900 is what per cent of $2\frac{1}{4}$?"

the equations

$$(a)\ \frac{R}{100} \times 900 = 2\tfrac{1}{4}$$

$$(b)\ 900 = 2\tfrac{1}{4} \times \frac{R}{100}$$

will produce the correct answers if there are no mistakes made in the work.

The other type presents less need for the use of the equation, as is shown by the following:

(*a*) "15 is 125% of what number?"
(*b*) "15 is $\frac{1}{8}$% of what number?"
(*c*) "15 is 75% of what number?"

The average pupil solves these without much trouble, although he may occasionally err in giving the answer to (*b*) as 120 instead of 12000. He may indicate his work by using the decimal equivalents for the per cents, thus:

$$(a)\ 15 \div 1.25 \quad (b)\ 15 \div .00125 \quad (c)\ 15 \div .75$$

or he employs fractions, thus:

$$(a)\ 15 \div 1\tfrac{1}{4} \quad (b)\ 15 \div \tfrac{1}{800} \quad (c)\ 15 \div \tfrac{3}{4}$$

Even in the foregoing, however, especially when they are mingled with examples in the other two cases, the habit of employing the equation is helpful. The problems are written in this way:

$$(a)\ 15 = \tfrac{125}{100} \times B \quad (b)\ 15 = \tfrac{1}{800} \times B \quad (c)\ 15 = \tfrac{75}{100} \times B$$

In solving an equation such as:

$$\frac{R}{100} \times 12 = 15$$

the first step is to combine the numbers of the first member into a single expression. This is done in two ways. The first form is an arithmetical one generally employed in French schools:

$$(a)\quad \frac{R \times 12}{100} = 15$$

From this the value of *R* is indicated thus:

$$(b)\quad R = \frac{15 \times 100}{12}$$

the multiplier (12) in the first member of (*a*) becoming a divisor in the second member of (*b*), and the divisor (100) in the first member of (*a*) becoming a multiplier in the second member of (*b*). The value of (*b*) is then obtained.

The other form omits the multiplication sign, as is done in algebraic equations, writing it thus:

$$\frac{12R}{100} = 15$$

The algebraic solution of this is worked in two ways, the first of which consists in the multiplication of both members by 100, which removes the fraction, giving the new equation:

$$12R = 1500$$

from which the value of R is obtained by dividing both members by 12, which gives:

$$R = 125. \quad 125\% \textit{ Ans.}$$

The other method consists in considering the equation to read $\frac{12}{100}$ times R equals 15, from which the value of R is obtained by dividing 15 by $\frac{12}{100}$, but inverts the fraction and writes it as a multiplier, $\frac{100}{12}$, thus:

$$R = 15 \times \frac{100}{12}$$

This last form follows the French one, and is considered more satisfactory than the algebraic method of "clearing of fractions," as it gives an opportunity of shortening the work by cancellation. A further discussion of the use of equations in arithmetic will be given in another chapter.

Pupils employing the equation method sometimes apply it to the first case; but this makes no particular difference. If they express

"What is 23% of 600?" in either of the following forms:

$$? = \frac{23}{100} \times 600 \quad \text{or} \quad B = \frac{23}{100} \times 600$$

the combination of the two terms in the second member gives the result, 138.

By the time the pupils reach the application of percentage in profit and loss and in other examples that involve the finding of the rate, they should be able to dispense almost entirely with the equation method of finding the latter, resorting to it only in cases of doubt. The value of this method of determining which of two given numbers is the base does not obtain in such examples as the following:

1. (*a*) "In 1900, a certain city had a population of 14,000; in 1910, its population was 15,000: what was the per cent of increase? (*b*) In 1900, a city had a population of 15,000; in 1910, the population was 14,000: what was the per cent of decrease?"

2. (*c*) "A man bought a piece of land for $3095 and sold it for $2476. What was his loss per cent? (*d*) What per cent is gained on land bought for $2476 and sold for $3095?"

After stating that the earlier population is taken as the base, some teachers have a pupil restate the first problem in these forms:

(*e*) "What per cent of the 1900 population, 14,000, is the 1910 population, 15,000?"

(*f*) "What per cent of the 1900 population, 15,000, is the 1910 population, 14,000?"

When the pupil obtains the answer to (*e*) as $107\frac{1}{7}\%$, and to (*f*) as $93\frac{1}{3}\%$, he can find the increase in (*a*) to be $7\frac{1}{7}\%$, and the decrease in (*b*) to be $6\frac{2}{3}\%$.

Problem 2 may be worked in the same way, with the preliminary explanation by the teacher that the cost is generally taken as the base, the pupil's restatement being as follows:

(*g*) "What per cent of $3095, the cost, is $2476, the selling price?"

(*h*) "What per cent of $2476, the cost, is $3095, the selling price?"

The answer to (*g*) is 80%, and to (*h*) is 125%, which gives the loss in (*c*) to be 20% and the profit in (*d*) to be 25%.

The employment of this type occasionally in a blackboard solution is useful in presenting in another way the fact that a gain of a certain per cent has a loss of a different per cent when the figures are reversed. This may be shown by the following problem:

"A man increased his capital of \$14,000 by $7\frac{1}{7}\%$, then lost $6\frac{2}{3}\%$ of his new capital. Is his present capital greater or less than the original one?"

(*a*)

$$\frac{14000}{14000}$$

$$\frac{15000 - 14000}{14000}$$

$$\frac{(15000 - 14000) \times 100\%}{14000}$$

$$\frac{1000 \times 100\%}{14000}$$

In working problem 1 many teachers recommend to their pupils that they first write the earlier population, the base, as a divisor, writing it also above the line; and as the next step that they write the later population above the line, before the earlier population if larger than the latter, or after it if smaller, indicating subtraction by the proper sign. The third step is to enclose these two given numbers in a parenthesis and to write outside of it 100% as a multiplier. The fourth step is to perform the subtraction and to remove the parenthesis. The first three steps, of course, require only one set of figures, the third of those shown in the foregoing illustration. The whole work takes this form:

(*b*)

$$\frac{15000}{15000}$$

$$\frac{15000 - 14000}{15000}$$

$$\frac{(15000 - 14000) \times 100\%}{15000}$$

$$\frac{1000 \times 100\%}{15000}$$

$$(a)\ \frac{(15000 - 14000) \times 100\%}{14000} = \frac{1000 \times 100\%}{14000} = ?\ \textit{Ans.}$$

$$(b)\quad \frac{(15000\ -\ 14000)\ \times\ 100\%}{15000} = \frac{1000 \times 100\%}{15000} = ?\quad Ans.$$

While many teachers require the work to be indicated, all do not appreciate the importance of beginning with the determination of the *base* as the chief consideration. This is made clear to the pupil by requiring him to write it before writing anything else.

$$(c)\quad \begin{array}{r} -\qquad\qquad \\ \hline \times\ 100\% \\ \hline 3095\qquad\quad \end{array} \qquad\qquad (d)\quad \begin{array}{r} -\qquad\qquad \\ \hline \times\ 100\% \\ \hline 2476\qquad\quad \end{array}$$

For convenience in performing the subtraction, some teachers prefer the following arrangement, problem 2 being used as an illustration. In this, the cost, $3095 in (*c*) and $2476 in (*d*), being the base, is first written below the second line as the divisor, the dollar sign being omitted; 100% is then written as a multiplier and the minus sign is located. The next step is to write the given numbers, the larger as the minuend, and to perform the subtraction.

$$\begin{array}{r} 3095 \\ -\ 2476 \\ \hline 619 \times 100\% \\ \hline 3095 \end{array} \qquad\qquad \begin{array}{r} 3095 \\ -\ 2476 \\ \hline 619 \times 100\% \\ \hline 2476 \end{array}$$

During the baseball season the records of the different clubs, as expressed in their per cents of victories, are of interest to boys. A problem may take the following form:

"What per cent of its games is won by a club that has won 27 games and lost 14?"

The operations may be indicated by placing the number of victories over the total number of games played and writing 100% as a multiplier.

$$\frac{27 \times 100\%}{27 +\ \ 14}$$

The per cent of defeats is indicated by writing their number above the line, etc. In these examples the base, which is the total number of games, is of subordinate importance, the first number to be written being that of the item (victories or defeats) whose per cent is required, generally the former.

$$\frac{14 \times 100\%}{27 + 14}$$

In solving a problem of this kind pupils should calculate the per cent of defeats also, using the sum of both per cents as a test of the correctness of the result. These are generally carried out to one decimal place, as given in the second column, .85 in the victories being changed to .9, the + after the 5 showing that the decimal is greater than $8\frac{1}{2}$, while the − after the other 5 shows that the second decimal is less than $1\frac{1}{2}$, for which reason 5 is rejected in the latter.

Victories	65.85 + %,	or 64.9% *Ans.*
Defeats	34.15 − %,	or 34.1% *Ans.*
Total	100.00 %	100. %

It may be advisable to call the attention of the pupils to the error in the customary newspaper reports in giving the foregoing as .649 under the heading of per cent, instead of giving it as 64.9, or changing the heading "per cent" to "rate." While it is necessary to carry out the rate to thousandths for many reasons, there is no excuse for using the decimal point in connection with the expression "per cent." In Europe .649 is written 649‰, the sign being read "per mil," meaning "thousandths" or "by the thousand."

Problems in finding the per cent that each of several items bears to the whole, are very practical, and should find a place in the percentage work in some simple form. The following is a typical example:

"In a school there are 24 pupils in the first class, 20 in the second, 18 in the third, and 16 in the fourth. What per cent of the pupils does each class contain?"

The results are as follows, the first column carrying them to three decimal places, the next to two, etc.

1st class	30.769%	30.77%	30.8%	31%
2d “	25.641%	25.64%	25.6%	26%
3d “	23.077%	23.08%	23.1%	23%
4th “	20.513%	20.51%	20.5%	20%
Total	100%	100%	100%	100%

In the last column the per cent of the 4th class is given as 20, although the decimal is greater than $\frac{1}{2}$, the rule being to increase only as many items as are necessary to make the total of 100, and to select the largest for such increase. The decimals in the per cents of the 1st and 2d classes, 30.8 and 25.6, being greater than that in the 4th, 20.5, the two former are made 31 and 26, respectively, and the latter is given as 20.

The introduction of percentage was due to the need of a common denominator in comparing numbers, especially in the form of fractions and ratios. Thus the relative size of $\frac{1}{3}$, $\frac{19}{60}$, and $\frac{7}{20}$ is not evident until they are expressed as $\frac{20}{60}$, $\frac{19}{60}$, and $\frac{21}{60}$, respectively. The relative size of 6 tenths, 61 hundredths, 605 thousandths, is not determined without an effort, especially when these numbers are heard merely. By changing them to 600 thousandths, 610 thousandths, 605 thousandths, their comparison is simplified. In order to have some one denominator that would be available for most purposes, 100 was taken; and as a further simplification, the term "per cent" was adopted for oral purposes, and the sign % in writing it. The use of the sign % is a vast improvement over the writing of 100 as a denominator, and it stands out more distinctly than the decimal point. When the per cents are not whole numbers they are written as mixed numbers or mixed decimals, the decimal places in the latter depending on the degree of accuracy required. The use of the common denominator, 1000, with its name *per mil* and its sign ‰, has made no headway in this country, although many things are based on the thousand, the death rate of a city, taxes in some instances, etc.

The different types of problems in profit and loss should not be taught separately, nor should profit and loss be made a separate subdivision of percentage. The following problem has already been given in other forms:

(*a*) "By selling a house for \$2665 a man makes a profit of $8\frac{1}{3}\%$. What was its cost?"

This corresponds with the following, which is not considered beyond the powers of some fifth-year pupils:

(*b*) "What number increased by $\frac{1}{12}$ of itself becomes 2665?"

The latter may be written in the form of an equation, using N to represent the number:

$$(c)\ N + \frac{N}{12} = 2665$$

which becomes, $\frac{13N}{12} = 2665$, etc.

In (*a*), the cost may be represented by C, giving rise to the following equation:

$$(d)\ C + \frac{8\frac{1}{3}C}{100} = 2665$$

which becomes, $\frac{108\frac{1}{3}C}{100} = 2665$.

In "analyzing" (*b*) the pupil says: "A number increased by $\frac{1}{12}$ of itself becomes $\frac{13}{12}$ of itself. When $\frac{13}{12}$ of the number is 2665, the number is 2665 divided by $\frac{13}{12}$, which equals 2665 multiplied by $\frac{12}{13}$." The equation form (*c*) follows the same procedure. The "analysis" of (*a*) is to the effect that 2665 represents 100% of the cost $+\ 8\frac{1}{3}\%$ of the cost, which is $108\frac{1}{3}\%$ of the cost, and that the cost is \$2665 divided by $108\frac{1}{3}\%$, or $\$2665 \div 1.08\frac{1}{3}$.

The following is a type of example that lingers in the textbooks:

"By selling two horses at $210 each, a drover gains $16\frac{2}{3}\%$ on one and loses $16\frac{2}{3}\%$ on the other. Find the net gain or loss on the two."

In obtaining the answer, many pupils find the cost of each horse, from which they ascertain the profit on one and the loss on the other, the difference between these two representing the net loss.

Their method is the following:

One horse is sold for $\frac{7}{6}$ of the cost and the other for $\frac{5}{6}$ of the cost.

7 sixths of cost is	$210	5 sixths of cost is	$210
1 sixth of cost is	$30	1 sixth of cost is	$42
The cost is	$180	The cost is	$252

The gain is $210 − $180, or $30. The loss is $252 − $210, or $42. The net loss is $42 − $30, or $12. *Ans.*

They overlook the fact that the second line, which gives one sixth of the cost, states the gain and the loss, respectively, and that there is no need of finding the respective costs.

The result might also be obtained without finding either the profit on one or the loss on the other, by considering that on one the loss is $\frac{1}{5}$ of $210 and the gain on the other is $\frac{1}{7}$ of $210; the net loss being $(\frac{1}{5} - \frac{1}{7})$ of $210, or $\frac{2}{35}$ of $210, which is $12.

In estimating their profits, business men do not go back to the cost. When their selling prices are fixed, for instance, at 25% above the buying prices, they assume that the profits are 20% of the receipts. This is, of course correct, since the receipts represent 125% of the cost, and a gain of 25% of the cost is $\frac{1}{5}$ of 125% – in other words 20% of the receipts.

This may be stated as follows:

$\frac{1}{1}$ of cost corresponds with $\frac{1}{2}$ of receipts
$\frac{1}{2}$ " " " " $\frac{1}{3}$ " "
$\frac{1}{3}$ " " " " $\frac{1}{4}$ " "
$\frac{1}{4}$ " " " " $\frac{1}{5}$ " "
$\frac{1}{5}$ " " " " $\frac{1}{6}$ " "
$\frac{1}{6}$ " " " " $\frac{1}{7}$ " "

Given as per cents:

100 % profit corresponds with 50 % of receipts
50 % " " " $33\frac{1}{3}$% " "
$33\frac{1}{3}$% " " " 25 % " "
25 % " " " 20 % " "
.20 % " " " $16\frac{2}{3}$% " "
$16\frac{2}{3}$% " " " $14\frac{2}{7}$% " "

Owing to the fact that they base their profits on the receipts, some business men disagree with the statements of the text-books that in profit and loss the cost is the base. They do not realize that when they are fixing their prices, they themselves actually start with the cost, increasing it by the appropriate rate.

Without generalizing as to the relation between profits and receipts, the teacher should give an occasional example that requires the finding of the per cent of the receipts constituting the gain.

The following is an oral problem:

"A boy sells newspapers at a profit of 100%. (*a*) What is his profit on papers for which he receives $1? (*b*) What per cent of his receipts does he gain?"

The following is a written problem:

"A merchant sold goods to the amount of $2675, on which his profit was 25%. (*a*) What was the cost? (*b*) What was the gain? (*c*) What per cent of $2675 did he gain?

(*d*) If his sales had amounted to $4500, what per cent of this sum would have been profit?"

The last question, (*d*), is introduced to show that the rate does not depend upon the amount of the sales.

Pupils find more or less difficulty in some types of problems expressed in general terms. For this reason it is useful to employ several such problems as the foregoing to prepare for the following:

(*a*) "When goods are sold at 125% of their cost, what fraction of the selling price is the profit?"

(*b*) "When goods are sold at an advance of 25%, what per cent of the selling price is the gain?’

The answer $\frac{1}{5}$ is more readily obtained in (*a*) than would be the equivalent 20%. The repetition of the term per cent is confusing to some when the statement is made that 20 per cent of 125 per cent is 25 per cent. The confusion is sometimes lessened by substituting the word "hundredths" for the first per cent, thus: 20 hundredths of 125% = 25%. Similar combinations will be discussed in connection with trade discount.

The peculiar feature of trade or commercial discount is the fact that when more than one discount is allowed, each successive discount is based upon the quantity remaining after the deduction of the previous discount. The simplest form of discount is the single allowance granted for a cash payment or for one made within a specified time, for instance, 10 days, when the bill is not due until the expiration of 30 days. The bills made out by pupils of the seventh grade should generally provide for a discount of this kind. In a bill of this type the pupil should write the discount at once as shown herewith. Rejecting .65 in the multiplicand, he multiplies 4 by 2, but carries 1, writing 9; then 3 by 2, writing 6; then 1 by 2, writing 2.

	$134.65
less 2%	2.69
	$131.96

The deduction of trade discounts of 25 and 10%, and a discount for cash of 2½%, is made in successive steps; that of 25% being obtained by dividing the total of the bill by 4; this is deducted and $\frac{1}{10}$ of the remainder is next written and subtracted; then $\frac{1}{40}$ of the second remainder. In finding $\frac{1}{40}$, the pupil divides by 4 and writes each quotient figure one place to the right of the corresponding figure of the dividend.

	$134.65
less 25%	33.66
	$100.99
less 10%	10.10
	$90.89
less 2½%	2.27
	$88.62

The teacher should consult the trade journals, which frequently quote prices and discounts. In some lines there are four and five successive discounts, and some large ones, such as 70, 10, 10, and 10%. The sign % is generally written with only the last of the series, to indicate that they constitute a single discount in a compound form.

Business men frequently find it necessary to compare two or more sets of discounts offered by competing salesmen, and they sometimes criticize a pupil that does not happen to know that 40 and 10% is not equivalent to 30 and 20%. A pupil generally prefers to begin with a sum of money, say $100, but 100% answers the same purpose. When the list price of an article is $100, a discount of 40 and 10% makes the net price $54, while a discount of 30 and 20% makes the net price $56, which shows that the first discount, 40 and 10%, is more favorable to the purchaser. The equivalent single discount for the first is 46%, and that for the second is 44%.

	100%
less 40%	40%
	60%
less 10%	6%
	54% net

	100%
less 30%	30%
	70%
less 20%	14%
	56% net

In comparing the two mentally, the business man finds the net value of the first to be 54% by subtracting each from 100% (100% − 40% = 60%; 100% − 10% = 90%) and finding their product (90% of 60%), 54%. The net value of the second is (100% − 30%) × (100% − 20%), or 70% of 80%, which is 56%. He finds the equivalent single discounts by subtracting each from 100% (100% − 54% and 100% − 56%). These are equal to the sums of the successive discounts given above, 40% + 6% in the first, and 30% + 14% in the second.

Problems in finding the "list" price when the net price and the successive discounts are given, correspond with some fractional types, such as:

(*a*) "After selling two-fifths of his sheep and three-tenths of the remainder, a drover has 42 sheep left; how many had he at first?"

This is of the same type as the following:

(*b*) "The net price of an article is $42 after the deduction of 40 and 30%; what is the 'list' price?"

The most direct solution of (*a*) is to consider that the final remainder, 42 sheep, is $\frac{7}{10}$ of the first remainder, which was $\frac{3}{5}$ of the original number. In other words, $\frac{7}{10}$ of $\frac{3}{5}$ of the original number, or $\frac{21}{50}$ of the original number, is 42, from which the original number is found to be 42 divided by $\frac{21}{50}$.

In solving (*b*), $42 is taken as 70% of 60% of the list price, or 42% of the list price.

Although successive discounts are generally stated in the order of their size, the largest first, they may be employed in any order in determining the net price. Thus, the net value of $80 less 33⅓, 25, and 10%, is found by first deducting 25% or 10%, since 80 is not exactly divisible by 3. If 33⅓% were

	$80.00
less 33⅓%	26.66⅔
	$53.33⅓
less 25%	13.33⅓
	$40.00
less 10%	4.00
	$36.00

	$80.00
less 25%	20.00
	$60.00
less 33⅓%	20.00
	$40.00
less 10%	4.00
	$36.00

first employed, three sets of fractions would occur during the work, which are avoided by the other plan.

Many of the sight drills recommended in the decimal work are available in this grade in the form of per cents. Products by $1\frac{1}{4}\%$, $2\frac{1}{2}\%$, 5%, $6\frac{1}{4}\%$, $6\frac{2}{3}\%$, $8\frac{1}{3}\%$, $12\frac{1}{2}\%$, 25%, $33\frac{1}{3}\%$, 50%, 125%, etc., are readily found by using the fractional equivalents $\frac{1}{80}$, $\frac{1}{40}$, $\frac{1}{20}$, $\frac{1}{16}$, $\frac{1}{15}$, etc.

Blackboard exercises with larger bases may employ the foregoing rates. These drills are of great help in retaining the facility of the pupils in combining numbers without the use of superfluous figures. It may be necessary for the teacher to prepare many of them to supplement the small quantity for which the text-book finds space.

CHAPTER III

INTEREST

Traditions seem to require the assignment of a much greater portion of the seventh-year work to the subject of interest than its mathematical importance warrants. The finding of interest is an ordinary exercise in the simplest form of percentage, with the introduction of a time condition.

The treatment of the subject in the sixth year is given in an earlier chapter. The new work in the seventh grade is the introduction of the time element as a compound number consisting occasionally of years, months, and days. The practice in this country of considering the year to consist of 12 months of 30 days each must be stated to the pupils, after which they may be permitted to work a few examples in their own way.

These examples should be of such a character that the portion of time expressed in days, or in months and days, should be reducible to simple fractions of a year; such as 180 da., 120 da., 240 da., 72 da., 144 da., 216 da., 36 da., 2 mo. 12da., 4 mo. 24 da., 7 mo. 6 da., etc.

To show the necessity for a general method, an example such as the following may be given:

"Find the interest on $372 at 5% for 2 yr. 7 mo. 23 da."

$$\begin{array}{rlcr}
 & \$372 & & \\
 & \underline{\quad .05} & & \\
12) & \underline{\$18.60} & \times\ \ 2 = & \$37.20 \\
30) & \underline{\$\ \ 1.55} & \times\ \ 7 = & 10.85 \\
 & \$\ \ .05\tfrac{1}{6} & \times 23 = & \underline{\ \ 1.19} \\
 & & \textit{Ans.} & \$49.24
\end{array}$$

To avoid the use of such a multiplier as $2\frac{233}{360}$, some pupil might hit upon the following method, in which he finds the interest for 1 year, then takes $\frac{1}{12}$ of this as the interest for 1 month, and $\frac{1}{30}$ of the latter as the interest for 1 day. The yearly interest is multiplied by 2, the monthly interest by 7, and the daily interest by 23, the sum of the three products being the interest for the given time, 2 yr. 7 mo. 23 da. The disadvantages of this plan are the inexact divisions that frequently arise, requiring the use of fractions in the quotient, or that the quotient be carried out to several decimal places. The custom of accountants to make division the last operation in calculations involving both multiplication and division is followed in the cancellation method. In this, the rate, 5%, is written as $\frac{5}{100}$ and the time, 2 yr. 7 mo. 23 da., is written as $\frac{953}{360}$, expressed as an improper fraction. This is obtained by adding 720 da., 210 da., and 23 da., and writing under their sum the denominator 360. After the other cancellations are made, the two remaining ciphers below the line are canceled and 953 is divided by 100 by placing a decimal point before the last two figures. The product of \$31 by 9.53 is obtained by using the former as the multiplier. The foregoing shows all the work.

$$\frac{\overset{31}{\cancel{\$372}} \times \cancel{5} \times 9.53.}{\underset{2\cancel{0}}{\cancel{100}} \qquad \underset{3\cancel{0}}{\cancel{300}}} = \frac{\$295.43}{6} = \$49.24 \textit{ Ans.}$$

$$\begin{array}{r} \\ 953 \\ 2859 \\ \hline 295.43 \end{array} \qquad \begin{array}{r} 720 \text{ da.} \\ 210 \\ 23 \\ \hline 953 \text{ da.} \end{array}$$

Even when the number of days in an example is reducible to a fraction of a month other than thirtieths, many teachers prefer to obtain the time in days, simplifying the fraction later in the cancellation. The reduction is made as shown above, the years being first expressed as days, then the months, etc., the average pupil finding no difficulty in writing directly the equivalents of 2, 3, 4, 5, etc., years. The employment of 100 as a divisor is usual, because of its assistance in the later so-called "problems."

This method should be developed by means of questions put to a pupil at the blackboard. When he is asked to indicate the work needed to obtain the interest, he probably does so in this way:

$$\$372 \times .05 \times 2\tfrac{233}{360}$$

In reply to further questions the reduction of the mixed number to an improper fraction follows:

$$\$372 \times .05 \times \tfrac{953}{360}$$

A hint as to the employment of common fractions throughout brings this:

$$\$372 \times \tfrac{5}{100} \times \tfrac{953}{360}$$

The directness of obtaining $\frac{953}{360}$ by the change of the given time to days and making the latter the numerator of a fraction of which 360 is the denominator will appeal to the pupil who has first expressed the time as $2\frac{233}{360}$ yr., and has changed this to the improper fraction.

Unless its use is required by the course of study, the six per cent method should not be taught, especially at the outset. Many pupils able to work examples rapidly in class by the latter method find considerable difficulty in calculating interest after they have been out of school a year or two. Their skill in obtaining the answers blinded their instructor to the fact that they did not thoroughly comprehend the reasons for their procedure, and that the formula was to them meaningless, one that they merely memorized for the time.

The six per cent method reverses the usual form of multiplying the yearly interest by the time in years, to the multiplication of the interest at 6% for the given time by the number of dollars in the principal. By each method, the value of the expression, $P \times R \times T$, is obtained by two multiplications. In the first, $(P \times R)$, the first product, is multiplied by T; in the second, P is multiplied by $(R \times T)$, the latter product being first determined.

By the 6% method, the interest on \$372 at 6% for 2 yr. 7 mo. 23 da. is ascertained as follows:

Interest on	\$1 for	2 yr.	=	\$.12
" "	\$1 "	7 mo.	=	.035
" "	\$1 "	23 da.	=	$.003\frac{5}{6}$
Int. for 2 yr. 7 mo. 23 da.				$\$.158\frac{5}{6}$

$$
\begin{array}{rr}
 & \$.158\frac{5}{6} \\
 & \times\ 372 \\
\hline
 & 6)1860 \\
\cline{2-2}
 & 310 \\
 & 316 \\
 & 1106 \\
 & 474 \\
\hline
\text{Interest at } 6\% & \$59.086 \\
\text{" " } 1\% & 9.847 \\
\hline
\text{Interest at } 5\% & \$49.24
\end{array}
$$

The interest at 5% is found by deducting from the interest at 6% one-sixth of itself. While the factor $.158\frac{5}{6}$ is obtained by simplifying $\frac{6}{100} \times \frac{953}{360}$, which indicates the product of the rate by the time, when the former is 6%, the mode recommended for the finding of this factor is not clear to some pupils. Moreover, the plan of beginnning with 6% and manipulating the first result to obtain the interest at the given rate is anything but apparent to many pupils of the 7th grade.

If a second method must be taught, the one by aliquot parts is the most useful in many respects. This consists in dividing the time into convenient portions, each of

			\$372	
			.06	
Int. for	1 yr.		\$22.32	
" "	1 "		22.32	
" "	6 mo.		11.16	$\frac{1}{2}$ of int. for 1 yr.
" "	1 "		1.86	$\frac{1}{6}$ " " " 6 mo.
" "	20 da.		1.24	$\frac{1}{9}$ " " " 6 "
" "	3 "		.186	$\frac{1}{10}$ " " " 1 "
			\$59.086	
		Ans.	\$59.09	

Principal	$372.	
Int. for 2 yr.	44.64	
“ “ 6 mo.	11.16	$\frac{1}{4}$ of int. for 2 yr.
“ “ 1 “	1.86	$\frac{1}{6}$ “ “ “ 6 mo.
“ “ 15 da.	.93	$\frac{1}{2}$ “ “ “ 1 “
“ “ 5 “	.31	$\frac{1}{3}$ “ “ “ 15 da.
“ “ 3 “	.186	$\frac{1}{10}$ “ “ “ 1 mo.
	$431.086	
Ans.	$431.09	

which is an aliquot part of a preceding one. When the amount is required, the multiplier is omitted, in order that the principal may constitute one of the addends. In obtaining the second result, the time has been divided differently, in order to show other possible aliquot parts.

Short methods in interest should be postponed until pupils are reasonably familiar with the general method. The former may then be introduced by oral examples which first combine the time and the rate into a simple factor. Thus, when the rate is 6%, the interest for 60 days, $\frac{1}{6}$ year, is 1% of the principal; that for 120 days is 2% of the principal, etc. In finding the interest mentally for 60 days at 6%, any principal may be used, the result being obtained by shifting the decimal point two places to the left. When the time is 120 days, 180 days, etc., making the multiplier 2%, 3%, etc., the principal should consist of such figures as do not render the calculation too difficult. Any numbers should then be used in blackboard exercises as principals, the pupils writing only the results, and the multipliers being any integral per cent to and including .12, the rate for 2 years.

After exercises have been given in finding how many days' interest at 5% will constitute 1% of the principal, 2%, 3%, 4%, 10%, examples similar to those given at the 6% rate should be used for oral, sight, and blackboard solution. The same procedure should be followed with rates of 3%, 4%, and 4½%.

The following are typical sight examples:
Find the interest on

(*a*)	$347	at	6 %	for	60 da.	*Ans.*	1%	of	$347
(*b*)	$483	“	5 %	“	72 “	“	1%	“	$483
(*c*)	$595	“	4 %	“	90 “	“	1%	“	$595
(*d*)	$617	“	$4\frac{1}{2}\%$	“	80 “	“	1%	“	$617
(*e*)	$754	“	3 %	“	120 “	“	1%	“	$754
(*f*)	$821	“	8 %	“	45 “	“	1%	“	$821
(*g*)	$712	“	6 %	“	180 “	“	3%	“	$712
(*h*)	$630	“	5 %	“	216 “	“	3%	“	$630
(*i*)	$520	“	4 %	“	270 “	“	3%	“	$520
(*j*)	$414	“	6 %	“	240 “	“	4%	“	$414
(*k*)	$320	“	$4\frac{1}{2}\%$	“	240 “	“	3%	“	$320
(*l*)	$505	“	4 %	“	180 “	“	2%	“	$505
(*m*)	$609	“	$4\frac{1}{2}\%$	“	160 “	“	2%	“	$609
(*n*)	$723	“	3 %	“	240 “	“	2%	“	$723
(*o*)	$832	“	8 %	“	90 “	“	2%	“	$832
(*p*)	$910	“	5 %	“	288 “	“	4%	“	$910
(*q*)	$230	“	8 %	“	135 “	“	3%	“	$230
(*r*)	$312	“	6 %	“	120 “	“	2%	“	$312
(*s*)	$420	“	5 %	“	144 “	“	2%	“	$420

If a pupil has difficulty in understanding the method, he may indicate the process on the blackboard, that for (*a*) being:

$$\$347 \times 6\% \times \tfrac{1}{6}\text{; that is, }\$347 \times \tfrac{1}{6}\text{ of }6\%\text{; or }1\%\text{ of }\$347.$$

In written exercises in finding the interest for periods of less than a year, the pupils may more profitably employ a modification of the aliquot parts method than the 6% one. Assuming that the interest on $743.50 is required for 144 days, at (*a*) 3%, (*b*) 4%, (*c*) $4\frac{1}{2}\%$, (*d*) 5%, or (*e*) 6%, the average pupil is more likely to understand a short method that employs the given per cent than one that begins with 6%.

At 3%, int. for 120 da.	$7.435	1% of principal
24 “	1.487	$\frac{1}{5}$ of int. for 120 da.
int. for 144 da.	$8.922	*Ans.* $8.92

At 4%, int. for 90 da.	$7.435	1% of principal
45 “	3.7175	$\frac{1}{2}$ of int. for 90 da.
9 “	.7435	$\frac{1}{10}$ of int. for 90 da.
Int. for 144 “	$11.896	*Ans.* $11.90

At 4½%, int. for 80 da.	$7.435	1% of principal
40 “	3.7175	$\frac{1}{2}$ of int. for 80 da.
20 “	1.8587	$\frac{1}{2}$ of int. for 40 da.
4 “	.3718	$\frac{1}{10}$ of int. for 40 da.
Int. for 144 da.	$13.383	*Ans.* $13.38

At 5%, int. for 144 da. is 2% of principal. *Ans.* $14.87

At 6%, int. for 120 da.	$14.87	2% of principal
24 da.	2.974	$\frac{1}{5}$ of int. for 120 da.
Int. for 144 da.	$17.844	*Ans.* $17.84

Pupils unable to understand this method should be permitted to employ the cancellation method or any other with which they are familiar. The aim should be rather to enable pupils to be accurate than to gain speed at the expense of accuracy. One advantage, however, of a second method, when it is really understood, is the opportunity it affords of testing a result by the employment of an entirely different set of figures.

In examples in finding the interest for a period intervening between two given dates, the teacher should ascertain the practice prevalent in the neighborhood. In some places the time is ascertained in days, when it is less than a year, and in years, months, and days, when greater than a year. In other localities, the time greater than a year is found in years and days. Even when it is obtained in years, months, and days, the second date may be such as to afford two different answers, according to the method employed.

1912 — 3 — 10
1909 — 11 — 13
2 — 3 — 27

Thus, the time between Nov. 13, 1909, and March 10, 1912, is found by compound subtraction to be 2 years 3 months 27 days. Another practice first finds the time between Nov. 13, 1909, and Nov. 13, 1911, as 2 years; the time between Nov. 13, 1911, and Feb. 13, 1912, as 3 months; and the time between Feb. 13, 1912, and March 10, 1912, as 26 days, making the entire time 2 years 3 months 26 days. A third method finds the time between Nov. 13, 1911, and March 10, 1912, to consist of 17 days + 31 days + 31 days + 29 days + 10 days, or 118 days, making the difference between the given dates 2 years 118 days, or 2 years 3 months 28 days, on the basis of 360 days to the year. Three different answers, each correct in some place, could be given in finding the interest between these two dates. To avoid confusing the pupils, only one method should be presented.

Most courses of study provide for the teaching of exact interest, which takes 365 days to the year. This is the only method used in England and in United States government calculations. Banks generally make interest payments on this basis, while collecting interest by the plan more favorable to them of the 360-day year. Fortunately for the pupils, exact interest does not lend itself to the 6% method; the result being obtainable only by finding the continued product of the principal by the rate by the time, the latter being either expressed, when it exceeds a year, in years and 365ths, or as an improper fraction.

In finding the exact number of days between two dates

Jan.	15 days
Feb.	29 "
Mch.	31 "
Apr.	30 "
May	31 "
June	30 "
July	31 "
Aug.	31 "
Sept.	4 "
	232 days

less than a year apart; for instance, between Jan. 16, 1912, and Sept. 4, 1912, one of the days is excluded, as a rule, and the other is included. In the accompanying illustration, the first day, Jan. 16, is omitted, the number of days remaining in this month being taken, 31 – 16. To this are added the number of days contained in each of the months to August, inclusive, and the number of days in September expressed by the date, thus including the last day.

As some places count both days, the teacher should ascertain the rule followed in the vicinity and employ it without referring to the other. The excuse for the method that includes both days appears to be the fact that a man borrowing money on the morning of April 15, and returning it during the afternoon of April 16, has had the use of it for portions of two days, and that he should accordingly be charged interest for this time.

Exercises in finding the interest between two dates are frequently given in connection with interest-bearing demand notes.

MARION, IND., NOV. 22, 1912.

On demand after date I promise to pay to the order of John Deshler, Four Hundred Dollars, value received, with interest at six per cent.

$\$400\frac{00}{100}$ ALEXANDER MCKINNEY.

If no payments have been made on this note, the amount due on Jan. 6, 1914, is $400 plus the interest from Nov. 22, 1912, to Jan. 6, 1914.

As a rule, lenders of money make provision in the note for annual or semi-annual payments of interest. If, however, these are not paid when due, the lender cannot ordinarily charge interest on the deferred payments. In a few states, however, a law has been enacted providing for the collection of simple interest on each deferred interest payment from the time it was due. This is called "annual interest," and is given a section in some of the text-books. The teacher should omit the topic unless the practice is legalized in the state in which his school is situated. An example involving annual interest is given on the next page.

In other states it may be necessary to give some exercises in partial payments, as this topic is apt to find a place in an examination paper. The first example may be based on the following note:

"CHICAGO, ILL., July 1, 1910.

"On demand, after date, I promise to pay to Alfred Topping, or order, One Thousand $\frac{00}{100}$ Dollars, value received, with interest at six per cent.

"\$1000$\frac{00}{100}$ GERALD TOFTE.

"Find the balance due July 1, 1914, when payments have been made as follows: July 1, 1911, \$300; July 1, 1912, \$300; July 1, 1913, \$300."

Face of note	\$1000.00
Int. July 1, 1910, to July 1, 1911	60.00
Amount July 1, 1911	\$1060.00
Payment July 1, 1911	300.00
Balance due July 1, 1911	\$760.00
Int. July 1, 1911, to July 1, 1912	45.60
Amount July 1, 1912	\$805.60
Payment July 1, 1912	300.00
Balance due July 1, 1912	\$505.60
Int. July 1, 1912, to July 1, 1913	30.336
Amount July 1, 1913	\$535.936
Payment July 1, 1913	300.00
Balance July 1, 1913	\$235.936
Int. July 1, 1913, to July 1, 1914	14.156
Amount due July 1, 1914	\$250.09 *Ans.*

The average pupil would have no difficulty in understanding that the amount due at the time of each payment should first be found, and the sum paid be then deducted, the remainder being taken as a new principal, etc.

The "catch" in an example in partial payments consists in having one or more of the payments smaller than the interest due at the time the payment is made. If the foregoing payments were $50 each, the amount due would be ascertained as follows:

In this case no payment being sufficiently large to cover the interest, the latter is calculated to the day of settlement, and from the amount at settlement the sum of the payments is deducted.

Face of note	$1000.00
Int. July 1, 1910, to July 1, 1914	240.00
Amount July 1, 1914	$1240.00
Payments $50 + $50 + $50	150.00
Due July 1, 1914	$1090.00

This procedure follows a decision of the Supreme Court of the United States to the effect that the law will not aid in the collection of compound interest in the absence of any statute permitting it. If the amount due were calculated each time a payment was made, the first balance would be $1060 – $50, or $1010; and the maker of the note would be compelled to pay interest on a sum $10 greater than the $1000 borrowed, thus being punished for making the payment; and this would occur with each payment, as is shown by the accompanying figures, which indicate a loss of $3.75 in making the three $50 payments annually instead of retaining them until July 1, 1914, which he may legally do, since the note does not specify the time when interest is payable.

$$\begin{array}{r}
\$1000.00 \\
+\ 60.00 \\ \hline
\$1060.00 \\
-\ 50.00 \\ \hline
\$1010.00 \\
+\ 60.60 \\ \hline
\$1070.60 \\
-\ 50.00 \\ \hline
\$1020.60 \\
+\ 61.236 \\ \hline
\$1081.836 \\
-\ 50.00 \\ \hline
\$1031.836 \\
61.9102 \\ \hline
\$1093.75
\end{array}$$

The amount due in a state permitting the collection of simple interest on deferred payments when the face of the note provides that interest shall be paid annually, would be $1093.60, the holder of the note being entitled to interest at 6% on $10 deficiency from July 1, 1911, to July 1, 1914; on $10 deficiency from July 1, 1912, to July 1, 1914; and on $10 deficiency from July 1, 1913, to July 1, 1914.

An example in partial payments should not be unnecessarily complicated. The intervals between payments should be made even months, the difference in time being so taken, and not expressed in days, which might involve the collection of 365 days' interest in a year, which would be an excess of 5 days when the year is assumed to contain only 360 days. The number of payments should be limited. If the exercises in the text-book are long, the teacher should replace them by shorter ones.

Unless called for by the course of study, the merchants' rule for ascertaining the balance due on an open account should be omitted. This is of such limited application that it should be taught only in the commercial course of a high school. This method charges interest on each debit from the time it falls due until the day of settlement, and allows interest on each credit from the time it is made until the day of settlement, the difference between the total of the debits (including interest) and the total of the credits (also including interest) showing the balance either way.

If compound interest is mentioned at all in the elementary school, its treatment should be of the simplest possible character. The only value of this topic is in connection with the determination of life insurance rates and the like. Savings banks do not pay compound interest according to the methods taught in the schools, since they allow interest, as a rule, only on even dollars, ignoring the cents in each principal.

The teacher must always be on guard against the introduction of unnecessary and unpractical topics, thus devoting time needed for arithmetical development to the discussion of unimportant matters.

When compound interest is a required topic, the pupil should not waste figures. To obtain the compound interest on \$747.50 for four years, at 6%, the rate need not be written. Each figure of the product by 6 is written two places to the right of the figure of the multiplicand producing it. The interest for each year is calculated on the amount at the end of the preceding year. The interest for 4 years is obtained by deducting the principal from the amount at the end of the 4th year.

Principal	\$747.50
Int. 1st yr.	44.85
Amt. 1st yr.	\$792.35
Int. 2d yr.	47.541
Amt. 2d yr.	\$839.891
Int. 3d yr.	50.3935
Amt. 3d yr.	\$890.2845
Int. 4th yr.	53.4171
Amt. 4th yr.	\$943.70
Principal	747.50
Int. 4 yrs.	\$196.20 *Ans.*

The so-called "problems" in interest can be taught in such a way as to be valuable training; it would be better to omit them entirely than to permit the pupils to work them by rules.

The latter generally provide for six types, which may be stated as follows, using the customary abbreviations:

I. Given I, R, and T; to find P
II. " A, R, " T; " " P
III. " I, R, " P; " " T
IV. " A, R, " P; " " T
V. " I, P, " T; " " R
VI. " I, A, " T; " " R

When the formula or rule method is employed, each type is considered separately, and the pupils work an example mechanically, many of them being content to obtain the answer without really understanding the reason for the various multiplications, divisions, etc.

By giving miscellaneous examples, no two successive ones being of the same type, and by requiring the pupils to obtain a result through the solution of an equation, a general method is employed that does not require the memorizing of half a dozen rules. While the equation method is not readily understood by pupils whose previous work has always been mechanical, it presents no particular difficulty to such as have been able to master the problems in fractions, decimals, and percentage, involving the same kind of reasoning.

The pupil's attention needs to be called only to the general formula, which he can readily state, even if he has not had it previously brought to his attention; that

$$I = \frac{P \times R \times T}{100}$$

Simple problems under types I, III, and V may be given to be expressed as equations:

I. "What principal will yield \$52.50 interest in 2 yr. 6 mo. at 6%?"

III. "In what time will \$350 yield \$52.50 interest at 6%?"

V. "At what rate will \$350 yield \$52.50 interest in 2 yr. 6 mo.?"

By means of appropriate questions, the average pupil should be able to write the following equations:

$$\text{I. } P \times \frac{6}{100} \times \frac{5}{2} = 52.50$$

$$\text{III. } 350 \times \frac{6}{100} \times T = 52.50$$

$$\text{V. } 350 \times \frac{R}{100} \times \frac{5}{2} = 52.50$$

The solution of each of the foregoing may be indicated at once in the following forms, if the teacher so desires:

$$\text{I. } P = \frac{52.50 \times 100 \times 2}{6 \times 5}$$

$$\text{III. } T = \frac{52.50 \times 100}{6 \times 350}$$

$$\text{V. } R = \frac{52.50 \times 100 \times 2}{350 \times 5}$$

and the result obtained by cancellation as follows:

I. $P = 350$. *Ans.* \$350
III. $T = 2\frac{1}{2}$. *Ans.* 2 yr. 6 mo.
V. $R = 6$. *Ans.* 6%

Some teachers may prefer to write the first member of each equation as a single expression, as follows, either reducing it to lowest terms or not:

$$\text{I. } \frac{30}{200} P = 52.50, \text{ or } \frac{3P}{20} = 52.50$$

$$\text{III. } \frac{2100}{100} T = 52.50, \text{ or } 21\,T = 52.50$$

$$\text{V. } \frac{1750}{200} R = 52.50, \text{ or } \frac{35R}{4} = 52.50$$

As decimals are not generally used in algebraic equations, others may prefer to employ instead of 52.50 its equivalent $\frac{105}{2}$ or $\frac{5250}{100}$.

These details are of comparatively slight importance, the chief aim at this time being to give the pupils, if they have not already acquired it, the habit of employing the equation in the solution of problems of any kind that may be simplified by its use. When the equation is correctly formed, the determination of the value of the unknown quantity is a simple matter.

Types II, IV, and VI differ somewhat from the others. They constitute the following problems:

II. "What principal will amount to \$402.50 in 2 yr. 6 mo. at 6%?"

IV. "In what time will \$350 amount to \$402.50 at 6%?"

VI. "At what rate will \$350 amount to \$402.50 in 2 yr. 6 mo?"

In IV and VI the interest, although not given directly, is readily obtainable by deducting the principal, \$350, from the amount, \$402.50. The equations may be stated as follows:

$$\text{IV. } 350 \times \frac{6}{100} \times T = 402.50 - 350$$

which is rewritten in the form given for III, the indicated subtraction being performed.

$$\text{VI. } 350 \times \frac{R}{100} \times \frac{5}{2} = 402.50 - 350$$

which may be rewritten in the form given for V.

$$\text{II. } P + \left(P \times \frac{6}{100} \times \frac{5}{2}\right) = 402.50$$

As a preliminary to the solution of this equation, the expression within the parenthesis must be simplified to either $\frac{3P}{20}$ or $\frac{30P}{200}$, being then stated as follows:

$$P + \frac{3P}{20} = 402.50 \quad (a)$$

The two terms of the first member may be combined into one, the resulting equation then being:

$$\frac{23P}{20} = 402.50 \quad (b)$$

from which the value of P is indicated

$$P = \frac{402.50 \times 20}{23}$$

etc.

The change in the form of equation (*a*) to that given in (*b*) is not new to pupils who have solved such a problem as either of the following:

(*a*) "What number increased by $\frac{3}{20}$ of itself equals $52\frac{1}{2}$?"

(*b*) "A man sells a cow for $52.50, thereby realizing a profit of 15%; what did he pay for the cow?"

Sight exercises in finding the principal, the rate, or the time should be postponed until the method of working the written ones has been mastered. The plan pursued in solving the sight exercises depends upon the numbers contained in them. The oral work that precedes and accompanies written work should be of such a character as to help to elucidate the method of doing the written work. Oral exercises that employ short methods are useful only after the mastery of the general method enables a pupil to give his whole attention to the shortest possible plan of getting the result without using a pencil.

CHAPTER IV

BANK DISCOUNT

In purchasing a note, a bank deducts the interest on the face of the note for the time it has to run, paying the remainder to the seller of the note. This advance deduction of interest is known as bank discount, and the remainder is called the *avails* or the *proceeds*.

An example in finding the bank discount presents nothing that has not already been taught in connection with interest. A few things, however, must be brought to the attention of the pupils.

In some states a note is not due at the time indicated, three additional days being allowed. Thus, a note calling for payment 30 days after date is not due in a few states until 33 days have elapsed; and in discounting this note, the bank deducts interest for 33 days if the note is discounted the day it is drawn. No allusion to days of grace should be made in states that have already abolished them.

The thoughtless pupil sometimes blunders in determining the time a note has to run. John Smith, for instance, may receive from William Brown a note for $600 dated April 1, 1912, payable in 90 days. The former may hold this note for some time before presenting it at a bank for discount. Assuming that he does so on June 17, 1912, the problem is stated as follows:

"Find the proceeds of a 90-days note for $600 drawn April 1, 1912, and discounted June 17."

The pupil who is accustomed in calculating interest to determine the time between the two specified dates ascertains the number of days between April 1 and June 17, which he finds to be 77. He accordingly calculates the interest on $600 at 6% for 77 days, from which he obtains the proceeds as $600 – $7.70, or $592.30. In such a case the teacher might ask what would be the proceeds if the note should be discounted on June 30. If the pupil deducts the interest for 90 days, making the proceeds $591, the teacher should inquire why a man would accept $591 from a bank when he could obtain $600 from the maker of the note. A few mistakes will show pupils that while the date on which the note is made is used in the determination of the date of maturity, the term of discount is the period intervening between the date on which the bank loans its money (in buying the note) and the day the note matures and the borrowed money is returned.

The teacher should give only one method of determining the time. He should ascertain how the banks in the vicinity calculate it, and omit all other considerations. If a text-book note is drawn and made payable in a state allowing days of grace, the teacher should ignore the latter if his own state does not employ them. If a note by its terms matures on Sunday or a legal holiday, the teacher should omit from consideration the fact that it may not be payable until a day or two later; he should use, as the date of maturity, the date that is the stated number of days after the date of the note.

The neighborhood practice should be followed in determining the maturity of a note for 3 months, as distinguished from a 90-days note. In most places the latter is due 90 days after it is drawn, while the former is due in 3 calendar months. Thus a 3-months note dated July 1, 1912, is due Oct. 1, 1912, and interest for 92 days is deducted if it is discounted on the first-mentioned day. On the other hand, a 90-days note made July 1, 1912, is due Sept. 29, 1912. If the banks in the vicinity of the school adopt a different method, the latter should be the only one used by the pupils.

The discount of an interest-bearing note is sometimes required. The pupils should be taught that the discount of a note is based upon the value of the note at maturity, which must first be determined. In an example to illustrate this, it is advisable to make the date of discount somewhat later than the day the note is drawn, especially when the interest rate and the discount rate are the same.

If a 120-days note for $1000 drawn May 1, and bearing interest at 6%, were presented for discount on May 1, the average bank would make the proceeds $1000 instead of $999.60, the latter being the ordinary result of their method. In the instance given the bank waives its 40 cents, to avoid making it too plain to a customer that the effect of its practice would be to charge interest for 120 days on the $20 interest included in the note. If the note were drawn for $1020, without interest, which is the amount of the first note at maturity, the proceeds would be $999.60.

Face of note	$1000.
Int. 120 da.	20.00
Amount at maturity	$1020.
Disct. 120 da.	20.40
Proceeds	$999.60

While there is no good reason for any objection to the collection of interest in advance in discounting a note, the text-books of a generation ago gave prominence to a type of problems headed "True Discount." They claimed that a man receiving $940 as the proceeds of a note for $1000, payable in a year, paid $60 interest on the $940 borrowed, which is greater than 6%. In their opinion the proceeds of this note should be such a sum as would amount in a year to $1000 at 6%; that is, $1000 ÷ 1.06, or $943.40.

This they called "present worth," the difference between it and the face of the note, \$56.60, being the true discount.

While there can be no objection to a problem of this kind stated in the proper form, it is unwise to add an unnecessary topic to an already over-full arithmetic course. "True discount" has no existence anywhere except in the text-books. Whenever a teacher finds it necessary to give such a problem, it may be stated thus:

"A man owes \$1000, payable in 60 days. How much should he deposit in a bank in order that the amount at 6% should be the sum he owes?"

In real life a man owing \$1000 in 60 days will nearly always find his creditor more than willing to accept \$990 in settlement, and frequently a smaller sum, the discount for cash sometimes being 1½% or 2% in such cases. Moreover, he might be unable to find a bank willing to pay 6% interest.

The teacher will do well to avoid the use of such expressions as "present worth" and "true discount" unless they form a part of the course of study.

Owing to the fact that the term of discount is generally 30, 60, or 90 days, and the rate is frequently 6%, pupils should be drilled through sight and blackboard exercises in the employment of the modification of the aliquot parts method recommended in finding interest for short periods.

Sight exercises might include the finding of the discount or the proceeds of:

I. \$375 for (*a*) 90 da. @ 4%, (*b*) 80 da. @ 4½%, (*c*) 72 da. @ 5%, (*d*) 60 da. @ 6%.

II. \$360 for (*a*) 45 da. @ 4%, (*b*) 40 da. @ 4½%, (*c*) 36 da. @ 5%, 30 da. @ 6%.

III. \$600 for (*a*) 30 da. @ 4%, (*b*) 20 da. @ 4½%, (*c*) 24 da. @ 5%, (*d*) 18 da. @ 5%, (*e*) 20 da. @ 6%, (*f*) 15 da. @ 6%, (*g*) 15 da. @ 4%, (*h*) 10 da. @ 4%, (*i*) 10 da. @ 4½%.

The blackboard exercises might employ the foregoing combinations of time and rate, the face of the note being any such sum as $1243.75, and only the discount being required.

In the written exercises, superfluous figures should be omitted. Some such arrangement as the following is suggested in finding the discount and the proceeds of, for instance, $843.50 for 63 days at (*a*) 4%, (*b*) 5%, (*c*) 6%, or for 90 days at (*d*) 4½%, (*e*) 5%, (*f*) 6%.

(*a*)	$843.50	(*b*)	$843.50
45 da.	4.2175	72 da.	8.435
+ 18 "	1.687	− 9 "	1.054
Discount, 63 da.	$5.90 *Ans.*	63 da.	$7.38 *Ans.*
Proceeds, 4%	$837.60 *Ans.*	5%	$836.12 *Ans.*

	(*c*)	$843.50
	60 da.	$8.435
	+ 3 "	.4217
Discount,	63 da.	$8.86 *Ans.*
Proceeds,	6%	$834.64 *Ans.*

(*d*)	$843.50	(*e*)	$843.50
80 da.	$8.435	72 da.	$8.435
+ 10 "	1.054	+ 18 "	2.109
Discount, 90 da.	$9.489 *Ans.*	90 da.	$10.54 *Ans.*
Proceeds, 4½%	$834.01 *Ans.*	5%	$832.96 *Ans.*

	(*f*)	$843.50
	60 da.	$8.435
	+ 30 "	4.217
Discount,	90 da.	$12.65 *Ans.*
Proceeds,	6%	$830.85 *Ans.*

When the course of study requires the finding of the face of note, term of discount, or rate, the pupils should be given an opportunity to attempt the solution for themselves. If they can manage the problems in interest, those in bank discount are not likely to prove very troublesome, especially when the equation method is employed. The following problems represent the types:

I. Face, \$240; discount, \$2.52; rate, 6%. Term?
II. Face, \$240; proceeds, \$237.48; rate, 6%. Term?
III. Face, \$240; discount, \$2.52; term, 63 da. Rate?
IV. Face, \$240; proceeds, \$237.48; term, 63 da. Rate?
V. Term, 63 da.; discount, \$2.52; rate, 6%. Face?
VI. Term, 63 da.; proceeds, \$237.48; rate, 6%. Face?

The discount, which is the continued product of the face of the note, by the rate expressed as hundredths, by the term (in years), may be indicated as follows:

$$D = F \times \frac{R}{100} \times T$$

Substituting in the foregoing the given values, the first problem gives rise to this equation:

I. $$240 \times \tfrac{6}{100} \times T = 2.52$$

from which the value of T is indicated

$$T = \frac{2.52 \times 100}{6 \times 240}$$

This may be solved in the manner shown below, canceling 100 and the decimal point in 2.52, etc. The result, $\frac{7}{40}$, indicates the time in years. As the answer is generally stated in days, the time, $\frac{7}{40}$ of 360 days, is given as 63 days.

$$\frac{\overset{\overset{7}{\cancel{42}}}{\cancel{252}}}{\cancel{6} \times \underset{40}{\cancel{240}}} = \frac{7}{40}$$

In II, the discount is obtained by subtracting the proceeds, $237.48, from the face, $240. The equation is then written as in I.

The following is the equation of III:

$$240 \times \frac{R}{100} \times \frac{63}{360} = 2.52$$

From this the value of R is indicated

$$R = \frac{2.52 \times 100 \times 360}{240 \times 63}$$

Canceling the decimal point and 100, etc., the value of R is found to be 6, the answer being 6%.

$$\frac{\overset{4}{\cancel{252}} \times \cancel{100} \times \overset{6}{\cancel{360}}}{\underset{\cancel{6}}{\cancel{240}} \times \cancel{63}} = 6. \quad \textit{Ans. } 6\%$$

Problem IV is solved by first finding the discount, as in II. The equation then becomes the same as that given in III.

The equation of V is the following:

$$F \times \tfrac{63}{360} \times \tfrac{6}{100} = 2.52$$

The value of F is thus indicated:

$$F = \frac{2.52 \times 360 \times 100}{63 \times 6}$$

Canceling, etc., the value of F is found to be 240.

$$\frac{\overset{4}{\cancel{252}} \times \overset{60}{\cancel{360}} \times \cancel{100}}{\cancel{63} \times \cancel{6}} = 240. \quad \textit{Ans. } \$240$$

Problem VI gives rise to the following equation:

$$F - (F \times \tfrac{6}{100} \times \tfrac{63}{360}) = 237.48 \quad (a)$$

Simplifying the expression within the parenthesis

$$F \times \frac{\cancel{6}}{100} \times \frac{\overset{21}{\cancel{63}}}{\underset{\underset{20}{\cancel{60}}}{\cancel{360}}} = \frac{21F}{2000}$$

the equation (a) then becomes

$$F - \frac{21F}{2000} = 237.48 \quad (b)$$

Subtracting $\frac{21F}{2000}$ from F, the equation (b) becomes

$$\frac{1979F}{2000} = 237.48$$

from which the value of F is indicated:

$$F = \frac{237.48 \times 2000}{1979} = \frac{474960}{1979}$$

$$\begin{array}{r} 240 \\ 1979\overline{)474960} \\ \underline{3958} \\ 7916 \\ \underline{7916} \\ 0 \end{array} \quad \textit{Ans.}\ \$240$$

Although the work could be shortened by cancellation, the fact that 23748 is a multiple of 1979 is not evident.

CHAPTER V

BUSINESS FORMS AND USAGES

The scope of instruction in business forms and usages depends upon the requirements of the course of study, and the wise teacher will not unduly extend it. The matter contained in the average text-book will usually cover the ground, and the reading of it in the class with a little discussion will probably be all that will be of much advantage to young pupils.

The teacher should, however, post himself as fully as possible as to business papers and accounts, in order to be able to give correct answers to inquiries made by the brighter pupils. There is a growing tendency among producers of all kinds to substitute calculations for guesses in the determination of the expenses of production, and there is more or less demand for instruction in some simple form of keeping accounts. The latter are frequently limited in text-books to store accounts with individuals, the debits being goods supplied, and the credits being cash or farm products.

When the farmer begins to feel the advisability of a more intelligent acquaintance with his financial progress, he begins with a simple form. This frequently leads to an account with each field, a dairy account, a poultry account, etc. Some find it advantageous to keep an account for each separate cow in a large herd, to ascertain which are unprofitable and should be disposed of.

The same is true of persons in other vocations, and the teacher is frequently called upon through his pupils to suggest suitable forms and also the items properly chargeable to either side of the account.

In all schools, pupils should be taught the proper method of making out a bill, or statement, beginning with the simplest form, which consists of a bill for goods purchased at one time. This may be followed in another grade by a bill for goods purchased at different times; later on, by one rendered by a mechanic for labor and materials, which are given in a previous chapter. In the upper grades the bills should show the deduction of the different discounts, and one form should consist of a statement of account, including both debits and credits. This last type should be preceded by a simple form of ledger account from which the statement is made.

These matters cannot be treated exhaustively in the limited time that is available for the subject. They more properly belong in the high-school, but it is inexpedient to neglect them altogether.

Receipts are next in order; there are two forms — receipts in full and receipts on account.

In connection with bank discount, the subject of banks, checks, and notes is occasionally called for by the course of study. If the text-book does not supply the necessary information, the teacher should procure it from other sources. He should, however, limit his treatment of all these matters to the capacity of 7th or 8th grade pupils.

Much of this work should be based upon the pupils' acquaintance with the various topics as determined by the teacher's questions. The differences between a savings bank and a bank of deposit need include only the ordinary ones of the allowance of interest by the former, and the employment of the check to draw money from the latter. Other details not covered by the text-book may well be omitted. Whenever a person becomes a depositor, he readily learns the use of the bank-book, check-book, stub, etc., whether he ever heard of them in school or not.

The treatment of promissory notes will depend somewhat upon the environment. Older pupils in agricultural sections are more likely to hear of notes than city pupils. Notices of sales at auction occasionally state that notes will be taken, etc. This form of note constitutes one type, a promise to pay at a specified time in the future a stated sum due for goods bought. The interest-bearing note is generally given as evidence of money borrowed. Such forms as joint notes, joint and several notes, etc., need not be unnecessarily dwelt upon, nor the various forms of endorsement. Unless required to do otherwise, the teacher should dwell upon these things only to the extent necessary to give meaning to the computations of interest, discount, etc.

In connection with the subject of bank discount, the pupils may be made acquainted with the two varieties of notes that are discounted. In some cases a merchant selling goods on credit receives a short-time note for the amount of his bill. If he desires to use the money before the note is due, he has it discounted at his bank. He endorses the note, thereby agreeing to pay it at maturity if the maker defaults.

The second type of note is one made by the person, usually a depositor, desiring to borrow money from a bank.

UTICA, N. Y., Nov. 14, 1912.

Sixty days after date I promise to pay to the order of Myself, Four Hundred $\frac{00}{100}$ Dollars, value received, at the First National Bank.

$400 $\frac{00}{100}$ JOHN WHALEN.

This note is indorsed by the maker, John Whalen, and also by some person willing to become security for its payment when due. The bank places the proceeds, $396, to the credit of Mr. Whalen, entering it in his bank-book. When the note becomes due, the bank charges $400 against Mr. Whalen's account, if the latter contains sufficient funds, and the note is returned when his balance is made up, together with the checks Mr. Whalen has drawn against his account.

Demand notes are frequently given as evidences of large loans, which are obtained from banks upon the deposit of securities. These are not endorsed, and are not discounted, the face of the note with interest being paid when the bank "calls" the loan.

The various forms of making payments in distant cities are sometimes discussed as a preliminary to the subject of exchange, foreign and domestic. Pupils may be asked to ascertain at the post-office, the express-office, or the telegraph-office the cost of making a transfer of $20, $50, or $100 by each of these agencies. The fact that larger sums are transmitted by bank drafts or cashiers' checks is then brought out, together with the comparatively small cost of the latter, as may be learned from the rates given in the newspaper quotations.

The use of the bank draft as a method of collecting money is sometimes a part of the required course, but it should receive only the briefest consideration.

Bills of exchange are important only in so far as they familiarize pupils with the relative values of the monetary units of the chief countries.

As the rates are given for 60-day bills, as well as those for sight bills, the element of discount does not enter. Drafts used in transmitting money throughout the United States being generally payable at sight, the cost of a draft is its face increased or diminished by a small sum. The quotations give the rate of premium or of discount per $1000; thus $1.50 premium means that a draft for $1000 will cost $1001.50; the rate of $1 discount means that the cost is $999.

CHAPTER VI

STOCKS AND BONDS

PRACTICALLY the only reason for the teaching of stocks and bonds is the fear of the teachers that a set of examination papers may have one or more questions on these topics. The fact that a person obtains the correct answer does not necessarily indicate the possession of any special mathematical ability; it means only that he has wasted time in getting hold of unimportant technical terms.

Finding the cost of 104 shares of stock at \$139.62½ plus 12½ cents per share; or the income from 125 shares at \$6 per share; or the rate of income when \$6 is received annually from a share costing \$194.37½ plus 12½¢, does not bring the pupil sufficient mathematical return to compensate him for taking time that could be devoted to more vital matters.

If the office of the school is to teach facts as well as figures, it might be well to have the former pretty nearly correct. If the pupils could get such a knowledge of the great dangers attending stock and bond investments as would enable them to avoid some of these pitfalls in their later lives, the time spent on those topics might not be entirely lost. If pupils could be made to understand that investments promising very much better results than the interest paid by savings banks are likely to end in the loss of the whole sum, the lesson might prove useful at a future time.

As most of the stocks have a so-called par value of \$100 per share, the teacher might require a pupil in his preliminary reading of a problem to express such prices as 124⅜, 212¾, etc., in terms of dollars and cents; \$124.37½, \$212.75, etc. The brokerage fee of $\frac{1}{8}\%$, being based on the par value, could be called 12½ cents per share; and a 5% dividend, which is also based on the par value, be stated as $5 per share.

It is probably inadvisable for the teacher to befog the matter further by explaining that while the $100 par per share possibly represents the cost to the first purchaser, it does not necessarily indicate that this sum has been spent for his benefit. It happens only too frequently that a large fraction of it is frittered away in advertising, in commissions to agents for disposing of the stock, in unnecessary salaries, etc., which makes the word "value" ridiculous. The number of shares of the par value specified in the certificate merely serves to show to what fraction of the assets and of the profits the owner of the certificate is entitled.

Thus, the owner of 200 shares in a corporation having a nominal capital of $600,000 divided into shares of $100 each has a one-thirtieth interest in the corporation, due to his ownership of 200 shares out of a total of 6000 shares. The $100 has no bearing further than to show by the number of times it is contained in the capital stock what fraction of the latter constitutes one share. The sum that has been paid per share, whether $5 or $500, has no particular relation to $100, although it is generally expressed as a per cent, but without the sign, being given as 5 or 500.

The quotations as generally stated in the papers are per cents. Thus, the price of Pennsylvania R. R. stock is given in New York as 130, which means $65 per share, the par value being $50. It is, however, quoted in the Philadelphia papers as 65, meaning $65 per share. Possibly only one other railroad stock has a par value other than $100.

There does not seem to be any satisfactory reason for assuming that the purchase of stock must be made through a broker, who receives 12½ cents per $100 share for his services. The subject would be simplified if each example stated the exact price per share paid by a purchaser or received by a seller, the fact that it was obtained through a broker, or directly from a private owner, having no particular bearing on the mathematical side.

Answers should, of course, not contain fractions of a share, a sum insufficient for the purchase of one share being stated as a remainder in money.

When problems give a par value other than $100 per share, the rate must be considered as a per cent, and the brokerage as $\frac{1}{8}\%$, to be added to the rate in finding the cost to a purchaser, or deducted in ascertaining the sum received by a seller.

The par value has no place in the determination of the rate of income, and may be neglected when it is stated. Some pupils, however, in finding the answer to a problem such as the following, seem to be helped by the introduction of $100.

"Which yields the better income, a 5% stock selling at 125 or a 3% stock selling at 80?"

That the respective rates are obtainable by dividing 5 per cent by $1\frac{1}{4}$ and 3 per cent by $\frac{4}{5}$ is not always understood.

$5\% \div 1\frac{1}{4} = 4\%$ *Ans.* $3\% \div \frac{4}{5} = 3\frac{3}{4}\%$ *Ans.*

The two-fold use of "per cent" is not always clear. That 5% of the par value is yielded by a share costing $1\frac{1}{4}$ times the same value may become clearer when it is stated that $5 interest is obtained annually from a capital of $125, from which the 4% rate is readily obtained, as well as the $3\frac{3}{4}\%$ rate from a yield of $3 from a capital of $80.

This difficulty in generalizing is experienced by many pupils throughout their percentage work, and they should be permitted to employ their own methods in solving problems such as:

(*a*) "A has 50% more money than B; what per cent has B less than A?"

(*b*) "A dealer marks his goods at 25% above cost and sells them at 25% below the marked price; what per cent does he gain or lose?"

(*c*) "What per cent is gained or lost by a person selling at the same price two articles, on one of which he gains 30% and on the other of which he loses 30%?"

(*d*) "What per cent of the selling price is gained on goods sold 12½% above cost?"

Problems in determining which of two stocks is the better investment are valueless to a business man, who knows that a dividend rate for a given year or series of years is not a certain indication of value. While the value is to some extent better shown by the selling price through a series of years than in any other way, even this is subject to a marked change due to various causes.

There is no need of going too deeply into the difference between stocks and bonds. The problems as to the purchase and sale of the latter specify their face value instead of their number with the par value of each. While there should be a reading and a discussion of the explanatory matter given in the text-book, any great elaboration is hardly worth while. The teacher should, however, take care that the pupils understand the statements.

Problems as to the income yielded by bonds should not be given. The life of a bond constitutes a very important factor, but one that cannot well be handled in the elementary school. A 4½% bond maturing in a year and selling at 97½ yields a much higher rate than the quotient of 4½% by .975, or $4\frac{8}{13}\%$. In this case, a $100 bond would cost $97½, the holder receiving during the year $4½ as interest and at the end of the year the face of the bond $100, the capital of $97½ thus amounting in a year to

$104½. This is equivalent, roughly, to $7 interest on a principal of $97½, making the rate, 7 ÷ .97½, or $7\frac{7}{39}\%$.

Of course, no well-secured 4½% bond maturing in a year is sold at 97½; this price is given merely to show how the time of maturity affects the rate of income.

Assuming that 101½ were paid for a like bond, the buyer would receive in a year $104½ as interest and principal on an investment of $101½, making the net interest for the year $3, which is 2.96% of $101½.

Other things being equal, bonds selling above par gradually diminish in price (and value) as the time of redemption approaches, while those selling below par increase, the value on the day of redemption being, in either case, 100.

One might infer from the text-book that there is practically no choice between the income from 6% bonds bought at 150 and 3% bonds bought at 75 when both have the same time to run. The business man would, however, in making a purchase, take into consideration the likelihood of a gradual drop in the price of one and a rise in the other.

Unless the text-book calls attention to the two forms of bonds, the registered and the coupon, it is hardly worth while to dwell upon the matter. The fact that a check for the interest due on the former is sent by mail to the person registered as the owner of the bond, while the holder of the other form cuts off semi-annually from each bond a coupon denoting the interest due, its date, place of payment, etc., does not necessarily benefit the arithmetical side. When a person makes his first bond investment, he learns all of these details without difficulty.

In finding the cost of bonds having a face value of $10,000, for instance, the rate being 137½ plus ⅛ brokerage, it is taken for granted that the purchase is made on an interest day, the seller receiving the interest then due. Assuming that interest on the foregoing bonds is payable May 1 and November 1 of each year, at $5\frac{1}{2}\%$, the buyer on November 1 would pay $13,762.50, the seller receiving also the half-yearly interest of $275, either in the form of coupons or by means of a check sent to him as the registered owner.

If the purchase were made on December 15, the buyer would be compelled to pay the interest accruing from November 1, 44 days, as he would receive on May 1 following, the half-yearly interest, which would include the interest for the 44 days during which the bond still belonged to the seller. This interest being $67.22, the cost to the purchaser on December 15 would be $13,829.72.

In order, therefore, to avoid bewildering pupils unnecessarily, no allusion is made in problems to the date of purchase, "accrued" interest, etc.

When stocks are purchased just before a dividend payment is due, the newspapers frequently quote prices "ex dividend," meaning that the seller obtains it. As the names of the owners of stocks are always registered in the office of the corporation, the dividend checks are sent to them. Changes of ownership are also registered; but for a few days before the dividend date, no changes are made, the registry books being then "closed." If a sale is made during this period, the seller receives the dividend. The price at this time is, however, a trifle below the ordinary one.

CHAPTER VII

RATIO AND PROPORTION

THE frequent employment of the terms ratio and proportion in all kinds of occupations renders it advisable to give the pupils some familiarity with their meaning, in connection with certain types of problems.

Mixtures of all kinds are made by farmers, mechanics, druggists, etc., by taking the component parts in certain ratios; and applications of many other kinds are very numerous. The pupils have employed ratios throughout the preceding grades without using the expression or being aware of the fact. When the cost of 3 articles is obtained by taking one-fourth of the cost of 12, the pupil has employed the ratio of 3 to 12, which is 1 to 4, or $\frac{1}{4}$.

The use of the colon to indicate division in many European countries simplifies the topic to some extent, the expression $3 : 12$ indicating both the ratio of 3 to 12 and the quotient of 3 divided by 12. In this country the latter is indicated as $3 \div 12$, the form $3 : 12$ being employed solely to indicate ratio.

The answer to the question "What is the ratio of 3 to 12?" is stated in the same form: "1 to 4." This result is obtained by considering the first-mentioned number as the numerator of a fraction of which the other number is the denominator, reducing the fraction to lowest terms, and giving the result as a ratio. The ratio of 12 to 3 is 4 to 1, the second term (1) being necessary to make the statement complete.

For purposes of comparison, concrete numbers are expressed in the same denominations, the latter, however, not appearing in the result, which should consist of abstract numbers. The ratio of 1 ft. 9 in. to 2 yd. is obtained by changing these terms to 21 in. and 72 in., respectively, the first fraction being $\frac{21}{72}$, which is changed to $\frac{7}{24}$, the ratio being expressed as 7 to 24.

While the ratio between two quantities is used a great deal in the solution of mental problems, it is not so useful in examples requiring the pencil. In the latter, the method of unitary analysis is generally applied to all types of problems involving only multiplication and division.

Thus, the cost of 3 articles at the rate of 80 cents per dozen may be ascertained (*a*) by finding the cost of 1 article and multiplying the result by 3, or (*b*) by multiplying 80¢ by the ratio of 3 to 12 expressed as a fraction. Both indicate the same operations, which are generally written in the form shown at (*c*). The oral solution takes the (*b*) form, since the ratio is a simple one. An example requiring the cost of 5 articles at 24 cents per dozen would employ the (*a*) form, since the cost of each is clearly 2 cents. In a written example with large numbers, the pupil saves time and worry by always employing the method shown at (*c*), and shortening the work by cancellation.

$$(a)\ \frac{80¢}{12} \times 3$$

$$(b)\ 80¢ \times \frac{3}{12}$$

$$(c)\ \frac{80¢ \times 3}{12}$$

A proportion is the expression of equality between two ratios. Thus, the ratio between 3 oz. and 1 lb. 4 oz. being the same as the ratio between \$1.20 and \$8, these four quantities may be written in the following form:

3 oz. : 20 oz. :: \$1.20 : \$8

the double colon (::) having the same meaning as the sign of equality (=). The expression is, however, read:

3 oz. is to 20 ounces as \$1.20 is to \$8.

The foregoing proportion might be the result of the following problem:

"Find the cost of a piece of silver weighing 1 lb. 8 oz. troy, at the rate of \$1.20 for 3 troy ounces."

This would be stated in this form:

$$3 : 20 :: (\$)1.20 : (\$)x$$

from which the value of x, the fourth term, may be ascertained by writing the product of the extremes as one term of an equation, of which the product of the means is the other; thus:

$$3x = 24$$

from which the value of x is found to be 8, making the answer \$8.

The foregoing does not show the cancellation, which should be employed to shorten the work.

In stating the foregoing proportion, the pupil is generally instructed to make the unknown number the last term, and the number of the same kind the third term, determining from the condition of the problem the order of the numbers constituting the other two terms.

Examples are occasionally given in which the required numbers constitute sometimes one term and sometimes another:

(*a*) $3 : 4 :: 60 : x$ (*c*) $6 : x :: 24 : 50$

(*b*) $7 : 12 :: x : 48$ (*d*) $x : 20 :: 36 : 80$

The reason for the given method may be shown by writing the foregoing proportions in the fractional form:

$$(a)\ \frac{3}{4} = \frac{60}{x} \quad (b)\ \frac{7}{12} = \frac{x}{48} \quad (c)\ \frac{6}{x} = \frac{24}{60} \quad (d)\ \frac{x}{20} = \frac{36}{80}$$

and supplying the missing number by inspection. In (*a*), for instance, the pupil sees that $\frac{3}{4} = \frac{60}{80}$, from which he notes that $3 \times 80 = 4 \times 60$, etc., and that $3x = 4 \times 60$.

Outside of its employment in the solution of problems in partitive proportion, this topic is little used in arithmetic work. In connection with the subject of proportion it is useful to call to the attention of pupils some such facts as the following:

(*a*) Although the interest on $1 for 2 years is twice as great as that for 1 year, the amount for 2 years is not twice as great as that for 1 year.

(*b*) That the chord of an arc of 180° is not three times as long as a chord of an arc of 60°, although the 180° arc is three times as long as the 60° arc in the same circle.

(*c*) That a boy able to walk 7 miles in 2 hours cannot walk 70 miles in 20 hours.

(*d*) That the cost of 2000 pounds of sugar is not necessarily 2000 times the cost of a single pound.

Partnership as given in arithmetics does not always correspond with actual business partnerships. If A and B engage in a business, A supplying $1000 towards the capital and B supplying $2000, the division of the profits giving $\frac{1}{3}$ thereof to A and $\frac{2}{3}$ to B would not be equitable when both devote their whole time to the partnership affairs. By the method usually given, a profit of $3600 for a year would give A $1200 and B $2400. If their services were equally valuable and the business was benefited $1500 a year by the work of each, a fairer plan would be to first allow $1500 to each, and then to divide the remaining $600 by giving A $200 and B $400, their respective shares then being $1700 and $1900. Another division might consist in allowing each 6% on his capital for a year, giving A $60 and B $120 and then dividing the remainder evenly, making A's share of the profits $1770, and B's $1830.

It not infrequently happens that a partnership agreement provides that one of two partners should receive one-half the profits without contributing any of the capital, his experience being considered an equivalent. Another arrangement between two such partners might provide for allowing interest on the capital to the one supplying it, next a reasonable sum to the other for his services and experience, the remaining profit to be shared according to agreement.

The text-book problems that base profits on the capital contributed by each multiplied by the time it continues in the business are interesting mathematically, but are rather far-fetched. The value of the services rendered generally receives as much consideration as the amount of capital invested.

CHAPTER VIII

LONGITUDE AND TIME

The extension of the employment of standard time throughout the world deprives the topic of solar time of any value except to the navigator. The time is the same throughout each country unless its extent east and west is particularly great. The time in many countries is exactly 1 hour, 2 hours, 3 hours, etc., faster or slower than Greenwich time. Ireland, however, employs Dublin time, and Russia employs the time of St. Petersburg. Other parts of eastern Europe have Greenwich time, Germany and middle Europe have time 1 hour later.

A problem is now very absurd which makes the time of a town in Spain, when it is 3 P.M. at Greenwich, depend upon the longitude of the former. The time of every place in Spain or France would be 3 P.M., unless the use of the 24-hour clocks in Spain might make more accurate the answer, 15 o'clock.

The maker of the old-time examination paper seldom neglected to include a question in longitude and time. Although the number of types was few, the opportunity to locate two places on different sides of the meridian, and to have one time before noon and the other after noon, furnished pitfalls for the unwary.

Such problems as the difference in latitude between two places on opposite sides of the equator, or the difference in longitude between a place east of the meridian and one west of it, are appropriate after the fifth year, while the difference between an A.M. hour and a P.M. one could be given even earlier. Oral problems asking the time difference between two places whose longitude difference is a multiple of 15° are not unreasonable. The combination, however, of all possible complications into a single problem, including the introduction of minutes and seconds of longitude and of time, has no longer any practical value except to the navigator, who calculates daily, at noon, the location of his vessel by noting the Greenwich time indicated by the ship's chronometer. He determines noon by an observation of the sun, and if the chronometer time is then 10:45 A.M., he knows that the vessel is 18° 45′ east of Greenwich, 15° for each hour of difference.

When the course of study calls for the teaching of longitude and solar time, a problem of this kind is not unreal and it states possible conditions. It requires the finding of the difference in longitude between two places, one of which is 0°, when the difference between a certain time and noon is given. The longitude of a place on land may be determined in a similar way, noon, for instance, at the place, being ascertained by a solar observation, which is compared with Greenwich time given by a chronometer.

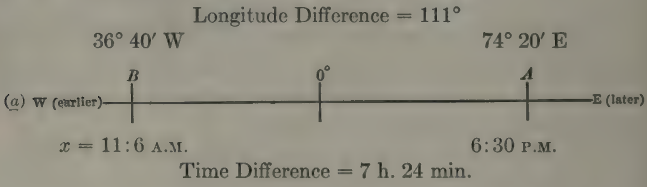
This topic belongs, however, with the study of latitude and longitude in a geography lesson.

When the course of study requires the solution of problems of this kind, the pupils will doubtless be aided by the use of a diagram in connection with each problem. Assuming that they have had the preliminary oral drills in the simpler types of finding time differences when longitude differences are given, and vice versa, the more complicated types of problems may be stated as follows:

	TIME AT A	TIME AT B	LONGITUDE OF A	LONGITUDE OF B
(*a*)	6:30 P.M.	?	74° 20′ E	36° 40′ W.
(*b*)	?	11:50 A.M.	70° 30′ E.	48° 30′ E.
(*c*)	9:20 A.M.	2:40 P.M.	?	20° 40′ W.
(*d*)	3:50 P.M.	10:10 A.M.	25° 20′ E.	?

Since latitude has no bearing on the time, the two places, *A* and *B*, may be located on a single east and west line. The eastern end is marked "later" and the western end "earlier," as an aid to the pupil in recalling the conditions.

The first step consists in drawing a line, writing E at the eastern extremity and W at the other. The words "Longitude Difference" are then written above, and "Time Difference" below. One of these is ascertained from the data given, and inserted in its place on the diagram; then the other is obtained by calculation and also inserted. The entire procedure is given in connection with the following solutions, one for each problem. The diagrams are, of course, not drawn to scale, the main object being first the proper location of *A* and *B* geographically, the more easterly of the two to the right, and secondly their situation with regard to the meridian, the latter falling between them or to one side, either the east or the west.

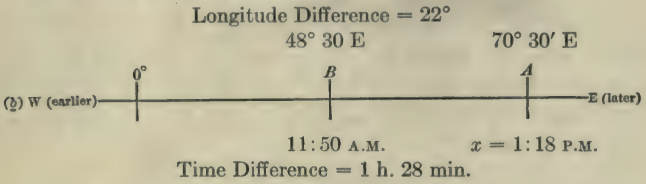


Locating *A* to the right, and marking it 74° 20′ E., the place of the meridian is roughly determined, and indicated by 0°, and to the west the point *B* is marked and its longitude written. Under *A* is given its time, and under *B* is written x, the time required. Above and below the line are written "Longitude Difference" and "Time Difference," respectively.

The solution begins with the insertion of the "Longitude Difference," which the pupil obtains by adding the given longitudes. From this is calculated the "Time Difference," which is then written. The pupil sees now that he must insert a time at *B* that will be 7 hours 24 minutes earlier than 6:30 P.M.; this is 54 minutes before noon, or 11:6 A.M.

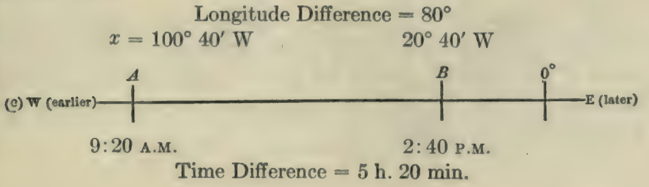
The use of more details in connection with the diagram is sometimes favored, in order to show all the steps employed; thus:

Longitude Difference = 36° 40′ + 74° 20′ = 111°
Time Difference = $\frac{1}{15}$ of 111 hr. = 7 hr. 24 min.
Time at *B* = 6:30 P.M. − 7 hr. 24 min. = 11:6 A.M.
These replace the ones given above.

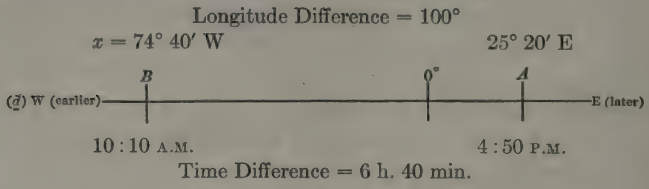


In (*b*) both places are east of the meridian. After each is located, its longitude noted, and the time at *B*, *x* is written under *A*, and the expressions "Longitude Difference" and "Time Difference" are also written. The value of the former is obtained, from which the latter is calculated. The last step is the addition of 1 hour 28 minutes to 11:50 A.M., to obtain the time at *A*, which is later than that at *B*, being more easterly.

In the third problem the times of *A* and *B* are first examined, from which *A* is found to be the more westerly, having earlier time.



After the insertion of the given data, the time difference is found and written, next the longitude difference, which shows the longitude of *A* to be 80° greater than that of *B*, or 100° 40′ W.



In (*d*) a look at the time shows that *B* is in west longitude; the 0° point is, therefore, placed between *A* and *B*. The procedure is the same as in (*c*) except that the longitude of *B*, which is 100° west of *A*, is obtained by subtraction; it is 74° 40′ west.

The difference in longitude is generally found by writing the hours, minutes, and seconds as degrees, minutes, and seconds, and multiplying the latter by 15. Thus, to find a longitude difference corresponding with a time difference of 4 hr. 12 min. 30 sec., the latter is written 4° 12′ 30″ and is multiplied by 15 as shown herewith. From a longitude difference of 63° 7′ 30″, a time difference of 4 hr. 12 min. 30 sec. is obtained by writing the longitude difference as 63 hr. 7 min. 30 sec., and dividing it by 15.

$$\begin{array}{r} 4^\circ\ 12'\ 30'' \\ \times\ 15 \\ \hline 63^\circ\ \ 7'\ 30''\ \textit{Ans.} \end{array}$$

$$\begin{array}{r} 15\overline{)63\text{ hr. }\ 7\text{ min. }30\text{ sec.}} \\ \hline 4\text{ hr. }12\text{ min. }30\text{ sec.} \end{array}$$

A few examples of both kinds may be used as blackboard exercises. The division type may be rather difficult, but pupils of this grade should be able to use 15 as a divisor. Some of them may be able to determine and employ a method of simplifying the work.

CHAPTER IX

SQUARE ROOT AND ITS APPLICATIONS

THERE are two general methods of explaining the process of extracting square root, the algebraic and the geometrical, one of which is generally given in the text-book, if not both. If the pupils acquire reasonable facility in the process, the teacher should not worry at their inability to master the reasons. The chief difficulty is likely to occur in the extraction of the roots of such expressions as .5, .125, 1.5, 101.125, etc., through mistakes in pointing-off. The first step in finding the square root of a decimal should, perhaps, be to make the number of decimal places even, by the annexation of a cipher when necessary.

As pupils find more interest in extracting a square root when it is connected with a problem, some teachers employ right triangles whose sides are expressed in integers. Any number greater than 2 can be one of the two sides forming the right angle. The other two sides of the triangle are obtained by separating the square of the given number into two factors, both even or both odd. One of these factors is taken as the sum of the other two sides and the other as their difference. One-half the sum of the factors will be the hypotenuse, and one-half their difference will be the remaining side.

Thus, when one side is 3, its square is 9, two factors of which are 9 and 1. One half of $(9 + 1)$ is 5, the hypotenuse; one half of $(9 - 1)$ is 4, the remaining side. When one side is 4, two even factors of its square are 8 and 2, one-half of their sum being 5, the hypotenuse, and one-half of their difference being 3, the third side. The factors 1 and 16 are not used, as one is odd and the other is even.

Some right triangles are the following:

One Side	Square	Factors	Hypotenuse (Half Sum)	Half Difference	First Side
3	9	9 × 1	5	4	3
4	16	8 × 2	5	3	4*
5	25	25 × 1	13	12	5
6	36	18 × 2	10	8	6**
7	49	49 × 1	25	24	7
8	64	32 × 2	17	15	8
		16 × 4	10	6	8**
9	81	81 × 1	41	40	9
		27 × 3	15	12	9**
10	100	50 × 2	26	24	10**
11	121	121 × 1	61	60	11
12	144	72 × 2	37	35	12
		36 × 4	20	16	12**
		24 × 6	15	9	12**
		18 × 8	13	5	12*
13	169	169 × 1	85	84	13
14	196	98 × 2	50	48	14**
15	225	225 × 1	113	112	15
		75 × 3	39	36	15**
		45 × 5	25	20	15**
		25 × 9	17	8	15*

Some of these triangles, marked *, have sides the same as others, while those marked ** have sides that are multiples of others. The foregoing comprise only 9 different triangles, but the number may be extended indefinitely.

Other applications are given in later chapters. These include the finding of the diagonal of a rectangle, the altitude of an isosceles triangle, the altitude or the slant height of a pyramid or of a cone.

The text-books give various methods by which the pupils may compare the area of the square described on the hypotenuse with the combined areas of the squares described on the other two sides. By drawing to a suitable scale the lines 3 and 4, or 8 and 15, meeting at a right angle, and finding that the hypotenuse of one measures 5 and the hypotenuse of the other measures 17, the pupil can to a certain extent, at least, compare the squares.

CHAPTER X

MENSURATION

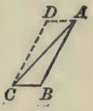
THE last year of the course generally includes a systematic treatment of the previous exercises in finding areas and volumes, together with its extension into general mensuration. The classification of plane surfaces into triangles, quadrilaterals, and other polygons, based upon the number of sides, is followed by that of triangles into their three varieties, equilateral, isosceles, and scalene, dependent on the equality or inequality of their sides. Another division of triangles depends upon the angles, triangles being also classified as acute-angled, obtuse-angled, or right-angled.

The quadrilaterals are also treated in the same way; those having the opposite sides parallel constituting the parallelograms, those having only two sides parallel constituting the trapezoids, and those having no sides parallel being the trapeziums. The equality of all four angles of a parallelogram makes the rectangle, which is called a square when the four sides are equal, and an oblong when two opposite equal sides are unequal to the other two. A parallelogram having two opposite equal angles unequal to the other two is called a rhombus when the four sides are equal, and a rhomboid when two equal opposite sides differ in length from the other two.

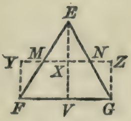
Besides naming the other polygons according to the number of their sides as pentagons, hexagons, heptagons, etc., no further designation is made except to denote a hexagon, octagon, etc., as regular when it is both equilateral and equiangular.

The attention of pupils may be called to the fact that although an equilateral triangle is necessarily equiangular, the same is not true of the other polygons, the rhombus, for instance, having its four sides equal but two of its angles being larger than the other two.

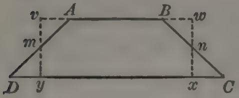
By means of questions the attention of pupils should be directed to the fact that the rectangle is the type from which the areas of the other polygons are developed. If the pupils have in the lower grades ascertained by cutting and superposition that the area of a parallelogram is equal to that of a rectangle having the same base and the same altitude, respectively, the review might be made by means of blackboard drawings. Similar means might be employed to show that the area of a triangle is one-half the product of its base by its altitude, by forming, a parallelogram, two adjacent sides of the triangle constituting two of the sides of the former, its remaining sides being formed by two lines drawn parallel to the first two. When DA, for instance, is drawn parallel to CB, and DC is drawn parallel to AB, the parallelogram $ABCD$ is double the triangle ABC.



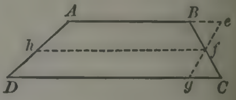
The triangle EFG may be changed to a rectangle by cutting along a line MN, through the middle points of two sides. The upper triangle EMN is divided into two parts by the perpendicular cut EX. By placing one part at the right of the remainder of the triangle, and another at the left, the rectangle $YZGF$ is formed, which is equal to the triangle EFG. The area of the rectangle being equal to the product of FG by YF, the area of the triangle is equal to FG by VX, VX being one-half the altitude of the triangle.



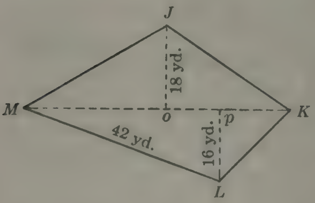
The trapezoid $ABCD$ may be changed into the rectangle $vwxy$ by cutting off the right triangles myD and nxC and



placing them in the positions shown by *mvA* and *nwD*, *m* and *n* being the center points of the sides *AD* and *BC*, respectively. The length of *xy* will be found to be one-half the sum of *AB* and *CD*. As the area of the rectangle is equal to *xy* multiplied by *vy*, the area of the trapezoid is equal, therefore, to the product of one-half the sum of the parallel sides by the perpendicular distance between them.



It may also be changed into the parallelogram *AegD* by drawing the line *hf* connecting the middle points of *AD* and *BC*, producing *AB* to *e*, making *Ae* equal to *hf*, and drawing *eg* parallel to *AD*. *Dg* (the base of the parallelogram) is equal to *hf*, and the altitude of the parallelogram is the altitude of the trapezoid. The middle line *hf* is the "average" of *AB* and *DC*, or one-half their sum. A trapezoid can therefore be transformed into a parallelogram having a base equal to one-half the sum of the parallel sides of the trapezoid and an altitude equal to its altitude.



The area of a trapezium, *JKLM*, for instance, is obtained by dividing it into two triangles by the line *MK*. The area of both triangles is the product of *MK* by one-half *Lp* plus *MK* by one-half *Jo*, or *MK* by one-half the sum of *Jo* and *pL*, the last two being perpendiculars let fall upon the diagonal *MK* from the vertices of the two angles opposite *MK*.

In the sight exercises in finding areas, opportunity should be taken to review the short methods of multiplying by aliquot parts of 100, etc. In getting the required products, the pupil should adopt such a plan as will be most suited to the combination in question. Thus, in finding the area of a trapezoid, one-half of the more convenient of the two factors may first be ascertained. When, for instance, the parallel sides are given as 21 and 30 and the perpendicular distance as 20, the operation takes this form:-

$$(21 + 30) \times \tfrac{20}{2};\ \textit{i.e.},\ 51 \times 10$$

When the parallel sides are 22 and 40 and the perpendicular distance is 13, this form is preferable:

$$\frac{22 + 40}{2} \times 13;\ \textit{i.e.},\ 31 \times 13$$

At times, the division by 2 may be left for the final step; as, for instance, when the parallel sides are 4 and 5 and the perpendicular distance is 7, it being simpler to find $\frac{1}{2}$ of 63 than to multiply 7 by $4\frac{1}{2}$ or $3\frac{1}{2}$ by 9.

Pupils of the eighth grade may reasonably be expected to do more in mensuration than merely to perform multiplication examples with given factors. Time that can be spared from bonds and stocks, longitude and time, and the like, might well be devoted to mensuration problems that require the pupil to determine one of the factors.

When a pupil knows that the base of an isosceles triangle is bisected by the perpendicular let fall from the apex, he should be able to find the area of an isosceles triangle whose base measures 6 feet and each of the equal sides 5 feet, first calculating the altitude, which is the perpendicular of a right triangle whose hypotenuse is 5 feet and whose base is 3 feet (one-half the base of the isosceles triangle).

Another similar problem might require the finding of the area and the perimeter of an isosceles triangle whose base is 6 ft. and whose altitude is 4 ft., each of the equal sides being the hypotenuse of a right triangle whose other sides are 3 ft. and 4 ft., respectively.

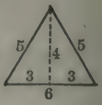
Although elementary text-books do not favor the method of finding the area of a triangle from the lengths of the sides, this is the only method available in some instances. It happens not infrequently that a person can measure the lengths of the sides of a triangular field without being able to locate and measure the altitude.

30
40
50
2)120
60 − 30 = 30
60 − 40 = 20
60 − 50 = 10
$\sqrt{60 \times 30 \times 20 \times 10}$
$= \sqrt{360000} = 600$
Ans. 600 sq. rd.

A figure known to be a right triangle might be selected for the first example employing this method, the sides being 30, 40, and 50 rods, respectively. The rule provides that from the half sum of the sides, 60, each side is subtracted in turn, leaving remainders 30, 20, and 10, respectively. The square root of the continued product of the half sum and the three remainders gives the number of square rods in the triangle.

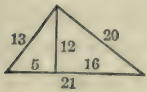
By comparing this number with the one obtained by finding one-half the product of 30 by 40, the pupils see that the results are alike, and that the first is, therefore, accurate.

5
5
6
2)16
8 − 5 = 3
8 − 5 = 3
8 − 6 = 2
$\sqrt{8 \times 3 \times 3 \times 2}$
$= \sqrt{144} = 12$
Ans. 12 sq. in.



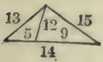
An isosceles triangle composed of two right triangles, each having sides, 3, 4, and 5 inches, may be used as a second illustration. The sides of the isosceles triangle are, respectively, 5, 5, and 6, and the area, as obtained from the length of the sides, is compared with the one resulting from multiplying the base by one-half the altitude.

If the teacher desires to obtain the sides of triangles having areas in whole numbers, he may form them by taking two right triangles, each having a side (not the hypotenuse) of the same length. In the illustration, one right triangle has sides of 12, 5, and 13, while the other has sides of 12, 16, and 20. Both together form a single triangle whose sides are 13, 20, and 21 the area of which is $\sqrt{27 \times 14 \times 6 \times 7}$, or 126.



The pupils are not informed of the method pursued in obtaining the sides of the scalene triangle whose area they are required to find. The teacher should, however, test the result obtained by the new method with the combined areas of the two right triangles employed.

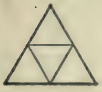
In another connection is given a method of calculating the altitude of the scalene triangles formed in this way.



A third scalene triangle has sides of 13, 14, and 15; a fourth has sides of 26, 51, and 55; and so on indefinitely: 17, 10, and 21; 17, 44, and 39; 51, 63, and 30; etc.

Some pupils may be able to understand the method of extracting the square root of $\sqrt{27 \times 14 \times 6 \times 7}$ by factoring, $\sqrt{(9 \times 3) \times (7 \times 2) \times (3 \times 2) \times 7}$; rearranging the factors, $\sqrt{9 \times (3 \times 3) \times (7 \times 7) \times (2 \times 2)}$; writing them $\sqrt{3^2 \times 3^2 \times 7^2 \times 2^2}$; which equals $3 \times 3 \times 7 \times 2$, the square root being extracted by writing the numbers with the exponents omitted.

The area and the altitude of an equilateral triangle are frequently required. The altitude of an equilateral triangle whose side is 1 is $\sqrt{1 - (\frac{1}{2})^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{.75} = .866$. The area of an equilateral triangle whose side is 1 inch is equal in square inches to the product of the base by $\frac{1}{2}$ the altitude; *i.e.*, $(1 \times .433)$ sq. in. = .433 sq. in.



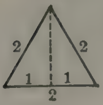
By requiring pupils to draw a 2-inch equilateral triangle, and to divide it into 1-inch triangles, and also to subdivide a 3-inch equilateral triangle in the same manner, they learn that the ratio between a 1-inch equilateral triangle and one having 2-inch sides is $1:2^2$, and that the ratio of a 1-inch equilateral triangle to a 3-inch one is $1:3^2$; in other words, that a 2-inch equilateral triangle has 4 times and that a 3-inch triangle has 9 times the area of a 1-inch triangle.

$$\begin{array}{r} 2\\ 2\\ \underline{2}\\ 2\overline{)6}\\ 3 \end{array}$$

$$\sqrt{3 \times 1 \times 1 \times 1} = 1.732$$

Ans. 1.732

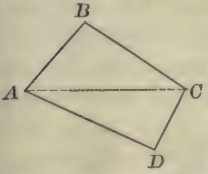
The fact that the area of a 2-inch equilateral triangle is four times .433 sq. in. may also be shown by calculating the area of the latter. This can be done by adding the three sides, etc., which gives the number of square inches in the area as $\sqrt{3}$, or 1.732. The same result is obtained by finding the altitude, $\sqrt{2^2 - 1}$ or $\sqrt{4 - 1}$, or $\sqrt{3}$. Multiplying this by 1, one-half of the base, the area in square inches is $\sqrt{3}$, or 1.732 sq. in.



The equivalent of the square root of 3 should be memorized. From it is derived the altitude in inches of a 1-inch equilateral triangle, $\frac{1}{2}\sqrt{3}$, or .866 in., and the area in square inches, $\frac{1}{4}\sqrt{3}$, or .433 sq. in. The altitude of an equilateral triangle whose side is 3, 4, 5, etc., inches is 3, 4, 5, etc., times .866 in., and its area is 9, 16, 25, etc., times .433 sq. in.

The diagonal of a square is the hypotenuse of an isosceles right triangle. When the sides of the square measure 1 inch, the hypotenuse in inches is $\sqrt{1 + 1} = \sqrt{2} = 1.414$. The pupils should obtain this result and memorize it. The length of the equal sides of an isosceles right triangle whose hypotenuse measures 1 inch should also be calculated. As the other sides are equal, $B^2 + P^2$ becomes $2B^2$, which is 1, making $B^2 = .5$, whence $B = \sqrt{.5}$, or .707. This length is one-half that of the diagonal of a 1-inch square.

If pupils are taught the method of finding the area of a triangle when the sides are given, the area of a quadrilateral may be obtained from the lengths of the four sides and the diagonal, the latter dividing the quadrilateral into two triangles. In problems of this kind, the sides and the diagonal must be so indicated as to enable the pupils to determine the two sides that, with the diagonal, constitute the three sides of the respective triangles.



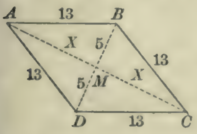
By combining two scalene triangles having one side in each of the same length, the latter will be the diagonal, and when the area of each triangle can be expressed in whole numbers, the work is not very tedious.

The following are examples:

(*a*) $AB = 26$, $CB = 40$, $CD = 39$, $AD = 45$, $AC = 42$
(*b*) " = 51, " = 63, " = 26, " = 28, " = 30
(*c*) " = 51, " = 55, " = 28, " = 30, " = 26
(*d*) " = 20, " = 21, " = 14, " = 15, " = 13
(*e*) " = 40, " = 42, " = 51; " = 55, " = 26

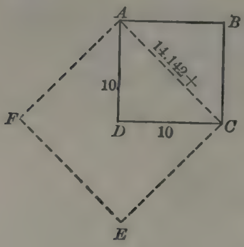
When pupils discover that the diagonals of a rhombus intersect at right angles, the area and the perimeter may be required of a rhombus whose diagonals are, for instance, 10 and 24, respectively.

The diagonals of a rhombus constitute its dimensions, since from them the area can be found. The length of the sides of a rhombus or of those of a rhomboid gives no indication of the area; neither do the diagonals of a rhomboid, since the latter do not intersect at right angles.



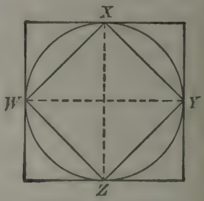
The area of a square *ACEF*, described on the diagonal of the given square *ABCD*, is a type of problem that appears in various forms.

As the area of *ACEF* is obtained by squaring *AC*, some pupils consider it necessary to ascertain the length of *AC*. This they do, when the side of the given square is 10, for instance, by adding 10^2 and 10^2, obtaining 200 for the square of *AC*. Not realizing that 200 is the required answer, they extract its square root, which gives 14.142+ for the length of *AC*, and square it, obtaining 199.996+, which is not entirely correct.

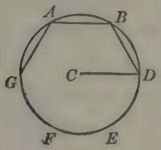


A variation of this problem consists in requiring the area of a square whose diagonal is 20 linear units. As 400, the square of the hypotenuse, represents twice the square of one of the equal sides, the square of a side is 200, which is the required area in square units. Unless the length of a side is asked, the square root should not be extracted.

In connection with the study of circles, a circle 7 inches in diameter may be inscribed in a square, and the circumference (22 in.), compared with the perimeter of the outer square (28 in.). Their respective areas, $38\frac{1}{2}$ sq. in. and 49 sq. in., may also be compared, the ratio in each case being 11 : 14. The area of a square inscribed in a circle may also be compared with that of the circumscribed square, the latter being double the former.



Another exercise is the construction of a hexagon. A circle is drawn, and the vertices of the hexagon are marked off on the circumference, the compass-points being kept the same distance apart as in the construction of the circle. The sides, *AB*, *BD*, *AG*, etc., are equal to *CD*, the radius.



By means of a circle enclosing an inscribed square and a second circle enclosing an inscribed hexagon, sight examples may be given to show that while the radius of a circle is equal to the chord of an arc of 60°, the diameter of a circle is the chord of an arc of 180°. When the diameter of a circle is 1, the chord of an arc of 60° is $\frac{1}{2}$, that of 90° is .707, and that of 180° is 1. The fact may also be brought out that the chord of 90° is also the chord of 270°, likewise that the chords of 60° and 300° are equal. The following illustrate these facts:

1. In a circle of $3\frac{1}{2}$ inches in diameter, give the length
   (*a*) Of the circumference (11 in.)
   (*b*) Of an arc of 180° ($5\frac{1}{2}$ in.)
   (*c*) Of the chord of an arc of 180° ($3\frac{1}{2}$ in.)
   (*d*) Of an arc of 60° ($1\frac{5}{6}$ in.)
   (*e*) Of the chord of an arc of 60° ($1\frac{3}{4}$ in.)
   (*f*) Of the chord of an arc of 300° ($1\frac{3}{4}$ in.)
2. When a circle is 2 inches in diameter, give the lengths of chords as follows:

(*a*) 180°, (*b*) 60°, (*c*) 300°, (*d*) 90°, (*e*) 270°.

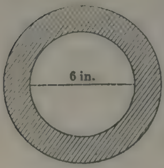
The length of the chord of 60° or of 300° is 1 inch, that of 90° or of 270° is the side of an inscribed square whose diagonal is 2 inches, the answer being 1.414 inches.

These examples and other similar ones are discussed again in the chapter on Construction Exercises.

Questions as to the ratio between the respective lengths of arcs of 60° and 180° will show that an arc containing three times as many degrees as another in the same circle is three times as long, although no such ratio obtains between their respective chords.

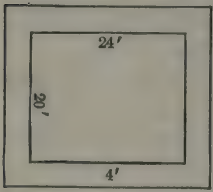
The ratio between the diameter of a circle and its circumference is generally taken as 7:22, the length of the circumference being obtained by multiplying the diameter by $3\frac{1}{7}$. This number is frequently expressed by the Greek letter π (pi). The value for π, $3\frac{1}{7}$, is also used in calculating the area.

In some examples the required steps should first be indicated. The surface of a circular frame 2 inches wide, and having an interior diameter of 8 inches, is obtained by diminishing the area of the large circle by that of the small one, the respective diameters being 12 inches and 8 inches. The area of the ring is indicated thus: $6^2\pi - 4^2\pi$. It is then expressed as $36\pi - 16\pi$, which equals 20π. The π is replaced by $3\frac{1}{7}$, the area being now indicated as 20 sq. in. × $3\frac{1}{7}$, making only a single multiplication necessary.

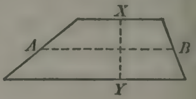


The same result may be obtained in another way by multiplying the average length of the frame by its width. The former is found by taking one-half of the sum of the outer and inner circumferences, 12π and 8π, their half sum being 10π. The product of this by the width, 2, gives 20π for the area, or 20 sq. in. × $3\frac{1}{7}$.

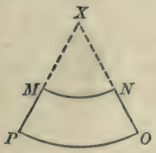
The area of a walk surrounding a rectangle may be found in the same way. When the walk is 4 ft. wide and the inner rectangle measures 24′ × 20′, the outer will measure 32′ × 28′. The average length of the walk is one-half the sum of the perimeters, which are 88 ft. and 120 ft., respectively. This length is $\frac{1}{2}$ of 208 ft., which is 104 ft. The area of the walk in square feet is, therefore, 4 times 104 sq. ft., or 416 sq. ft.



While the result in this case is more readily ascertained by diminishing the outer area, 896 sq. ft., by the inner area, 480 sq. ft., the foregoing is given to show the method frequently employed by mechanics, who



adopt the same plan in ascertaining the area of a trapezoid, the "average" length of which, AB, is equal to the half sum of the parallel sides, as was shown on a preceding page.



The area of the surface $MNOP$, which may be considered a portion of a circular frame, is found by multiplying its "average" length $[\frac{1}{2}(MN + PO)]$ by its width, MP or NO. The result could also be obtained by deducting the area of the sector XMN from that of XPO, but this would require the preliminary determination of one radius XM (or XP), from which the other could be derived by adding (or subtracting) MP.

After the pupil has learned the following formula for the area of a circle:

$$A = \pi R^2$$

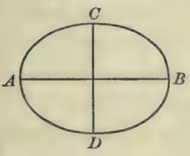
he should not burden his memory with one for finding the diameter or the radius when the area is given. If the latter is 154 sq. in. for instance, the pupil writes the following equation:

$$3\tfrac{1}{7}\, R^2 = 154$$

from which he obtains the value of R^2 by dividing 154 by $3\frac{1}{7}$:

$$R^2 = 154 \div 3\tfrac{1}{7}$$

From this he finds the value of R^2 to be 49, whence $R = 7$.



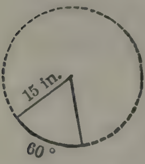
The area of an ellipse is obtained by multiplying the product of the semi-axes by $3\frac{1}{7}$. Thus, where AB is 10 in. and CD is 6 in., the area is

$$3\tfrac{1}{7} \text{ times } (5 \times 3) \text{ sq. in.}$$

The area of the ellipse has the same ratio to that of the circumscribing rectangle as the area of the circle has to that of the circumscribing square, the ratio being in each case 11 to 14. The fraction $\frac{11}{14}$ is $\frac{1}{4}$ of $3\frac{1}{7}$.

While the circumference of the circle bears a like ratio to the perimeter of the square, there is, however, no such relation between the bounding curve of the ellipse and the perimeter of the circumscribed rectangle.

The area of a sector can be found from the radius of the circle and either the number of degrees contained in the arc or the length of the arc.



In the former case, the area of the circle is first indicated, then the area of a sector of one degree is indicated by placing 360 under the former as a divisor, and that of the given sector by multiplying the quotient by the number of degrees contained in the arc. Thus the area of a sector of 60°, in a circle whose radius is 7 inches, is indicated thus:

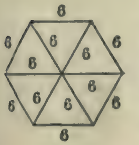
$$\frac{(7^2 \times 3\frac{1}{7}) \text{ sq. in.} \times 60}{360}$$

In this case it is $\frac{1}{6}$ of the area of the circle; that is, $\frac{1}{6}$ of 154 sq. in., or $25\frac{2}{3}$ sq. in.

When the length of the arc is given, the area is equal to $\frac{1}{2}$ the product of the radius by this length. In the foregoing circle, the circumference is 22 inches, which makes the arc of the circle $3\frac{2}{3}$ inches long; 7 times $3\frac{2}{3}$ gives the number of square inches, $25\frac{2}{3}$.

The names *quadrant* and *sextant* for sectors of 90° and 60°, respectively, may be used by the teacher as occasion requires.

Pupils able to find the area of an equilateral triangle need no rule for the determination of the area of a regular hexagon, the latter being made up of six equilateral triangles. When each side of the hexagon measures 1 inch, its area is 6 times .433 sq. in., or 2.598 sq. in.; when each side of the hexagon measures 2 inches, its area is 6 times 1.732 sq. in.; 433 sq. in. being the area of a 1-inch equilateral triangle and 1.732 sq. in. being the area of a 2-inch equilateral triangle.



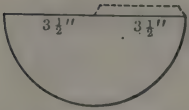
The surface of a right regular pyramid and that of a right cone have been presented already in another way, the convex surface of the pyramid consisting of a series of equal isosceles triangles, and that of the cone corresponding with the area of a sector. When the entire surface of a pyramid or of a prism is required, the base should be limited to one of the polygons already studied.

The convex surface of the frustum of a right regular pyramid is made up of a number of equal trapezoids, and that of a right cone is a part of a circular disk. Each is obtained by multiplying the average of the upper and the lower perimeter by the slant height.

In the study of surfaces and volumes the pupil receives considerable assistance by making hollow models of cardboard, by molding solid models in clay, or by cutting them out of turnips, etc.

The relation of the volume of the cone to that of the cylinder may be illustrated by making a cardboard model of each, the base and the altitude being the same in both. By filling the cylinder with sand, using the cone as a measure, the number of conesful required will indicate the respective volumes. For this purpose the cone should have no base, and the cylinder no top.

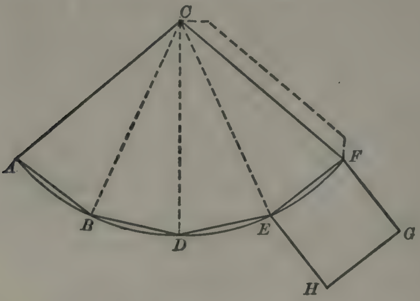
A sector of any convenient size, with a flap for pasting, is used for the cone, which should first be made. In the



accompanying illustration a semicircle is shown. When the cone is formed, the diameter of its base is measured, and its altitude, and a cylinder is constructed having these dimensions.

A hollow-square pyramid is made by drawing an arc having a radius equal in length to that of one of the oblique edges. On this arc are laid off four successive chords, each equal in length to a side of the base.

In the accompanying figure an arc is drawn with C as a center and a radius AC, the length of an oblique edge.



Equal chords are drawn, AB, BD, DE, and EF, forming, when folded, the sides of the square base. A pasting-flap may be left on CF.

If the school contains sets of drawing models, the weight of three equal wooden cones may be compared with that of a cylinder having the same dimensions. The sphere in these sets has generally the same diameter as the cylinder, but the height of the latter is twice its diameter. If three of the spheres weigh as much as one such cylinder, the weight of each sphere is one-third of that of the cylinder, and two-thirds the weight of a cylinder having an altitude equal to the diameter of the sphere.

The respective volumes of the pyramid and the prism may also be compared by weighing the models.

While the greater part of the work in solids should consist in the finding of surfaces and of volumes when the required dimensions are given, an opportunity should be afforded the brighter pupils to solve problems in which one or more of the dimensions must be calculated. Some types are contained in the following:

(*a*) Find the slant height of a right square pyramid, the altitude being 12 ft., and one side of the base 10 ft.

(*b*) Find the altitude of a right pyramid having a slant height of 17 inches, and a base 16 inches square.

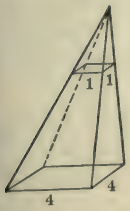
(*c*) Find the two slant heights of a right pyramid whose base is a rectangle 10 in. by 18 in. and whose altitude is 12 in.

(*d*) Find the height of a square prism whose convex surface is 160 square feet and whose entire surface is 192 square feet.

(*e*) Find the volume of a square prism whose convex surface is 160 sq. yd. and whose entire surface is 192 sq. yd.

The exercises in finding the surfaces and volumes will suggest many other types of more or less complexity.

The volume of the frustum of a pyramid or of a cone should not be required unless it is prescribed by the course of study. The correctness of the rule given in the text-book may be shown by considering that from a pyramid 12 in. high and having a base 4 in. square, a portion 3 in. high is cut off. If the pupils understand that this portion, having one-fourth of the height of the original pyramid, has a base each of whose sides is one-fourth that of each side of the lower base, they can find the volume of the frustum by deducting the volume of the part cut off from that of the



whole. That is, its volume in cubic inches is $\frac{1}{3}(12 \times 4^2) - \frac{1}{3}(3 \times 1^2) = 64 - 1 = 63$.

The rule for obtaining the volume of a frustum is to find the product of one-third of its altitude by the sum of the area of the upper base + that of the lower base + that of a mean proportional between the bases; or, in this case,

$$\begin{aligned} & \tfrac{9}{3} \times (1 + 16 + \sqrt{16 \times 1}) \text{ cu. in.} \\ = {} & \tfrac{9}{3} \times (1 + 16 + 4) \text{ cu. in.} \\ = {} & (3 \times 21) \text{ cu. in.} \\ = {} & 63 \text{ cu. in.} \end{aligned}$$

By the mean proportional between the areas is meant the square root of their product. When the bases are squares, the mean proportional is the product of the side of one square by the side of the other. In the given example, it is 4 sq. in. When the sides are 3 and 5, their areas are 3^2 and 5^2, and their mean proportional is $\sqrt{3^2 \times 5^2}$, which is 3×5.

The mean proportional of the two bases of the frustum of a cone is $3\frac{1}{7}$ times the product of the two radii; that is, if the radius of one is 3 and that of the other is 5, the area of one is $3^2\pi$, of the other is $5^2\pi$, and of the mean proportional is $\sqrt{3^2\pi \times 5^2\pi}$, or $3 \times 5 \times \pi$. The combination of these three base-areas is indicated thus: $(3^2 + 5^2 + 3 \times 5)\pi$, which equals $(9 + 25 + 15)\pi$, or 49π. Their product by one-third of the altitude, when the latter is 4, for instance, is thus indicated, π being replaced by its equivalent, $3\frac{1}{7}$:

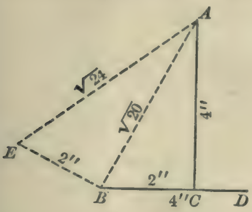
$$49 \times 3\tfrac{1}{7} \text{ by } \tfrac{4}{3};$$

$$\text{that is, } 49 \times \frac{22}{7} \times \frac{4}{3} = \text{Volume.}$$

It would be absurd to find the area of the upper base of the frustum by multiplying 9 by $3\frac{1}{7}$, and that of the lower by multiplying 15 by $3\frac{1}{7}$, then finding the product of these two areas, extracting the square root, etc.

It is sometimes desired to construct a hollow paper pyramid having given dimensions, as, for instance, a base 4 inches square and an altitude of 4 inches. Instead of calculating the length of an oblique edge, it may be determined thus:

Draw *BD*, 4 inches. At its middle point, *C*, draw a perpendicular 4 inches high, using a rectangular card. The line *AB* shows the length of the slant height. By means of the card draw *EB*, 2 inches long, perpendicular to *AB*; the line *AE* represents the length of an oblique edge. By employing *AE* as the radius of a circular arc, laying off four chords of four inches each, cutting and folding, the required pyramid will be obtained.



To make a hollow cone with a given base and a given slant height, the number of degrees in the arc of the required sector must be determined, the radius of the sector being the given slant height. When the diameter of the base and the slant height are equal, say 4 inches each, the circumference of the base is 4π inches, and the arc of the sector must have the same length, 4π inches. As the radius is 4 inches, the circumference of the circle containing the sector is 8π inches. The arc of 4π inches being one-half of the circumference, the required sector is a semicircle.

The ratio between the arc of the sector and the circumference is the same as the ratio between the radius of the base and the slant height of the cone; in the foregoing example this is 2 to 4, or one-half.

For a cone having a slant height of 5 inches and a base with a radius of 2 inches, the radius of the sector is 5 inches and the arc measures $\frac{2}{5}$ of 360°, or 144°. Two 5-inch lines making an angle of 144° are drawn by means of the protractor described in the chapter on Geometrical Constructions.

When the altitude is given, the ratio between the slant height and the radius of the base is not always a simple one. In such a case the number of degrees in the angle may be determined as nearly as possible by calculation, and then laid off by means of the protractor.

FIFTH PART — APPENDIX

CHAPTER I

EQUATIONAL ARITHMETIC

THERE is a growing tendency towards the introduction of the equation into the arithmetic course of the last two years, substituting it for many of the useless applications of percentage and for some obsolete topics. This new subject should by no means take the form of a completion of a definite portion of high-school work, in order to diminish the amount of algebra to be taught in the ninth year. Teachers who follow the order of the regular text-book in algebra, devoting time to the fundamental operations, including work in algebraic fractions, and giving much practice in finding factors, do not afford their pupils the chief advantages to be derived from the new branch. Pupils who do not go on to high-school algebra fail to receive the benefit that could be obtained from practically exclusive attention to the solution of problems by means of the equation.

All other topics should be entirely subordinate and incidental, being studied only to the extent required by the main object. The work in addition need be carried only so far as to enable a pupil to combine $6x$, $3x$, and $-2x$, for instance. That in subtraction should not extend beyond the removal of a parenthesis containing an expression of two or three terms, and preceded by a minus sign. The early work in multiplication should be limited to the combination of two such factors as $3x$ and 7, the resulting product being used as a dividend and the arithmetical factor as the divisor in the only division examples that are necessary. Fractions require no special treatment, an equation containing them being simplified by the method known as "clearing of fractions."

Algebra, properly so-called, has no place in the elementary school; and when it is taken up in the high school by pupils who have solved equations in the grammar grades, the teacher should begin at the beginning, employing the scientific treatment properly belonging to the formal study of this branch.

The work in equations in the seventh or eighth grade should not commence with definitions, literal quantities, or the like. A beginning may be made by asking answers to the following, which can be given by a third-year pupil:

(*a*) $3 + ? = 7$
(*b*) $? - 6 = 9$
(*c*) $8 - ? = 2$
(*d*) $? + 5 = 6$
(*e*) $3 \times ? = 6$
(*f*) $9 \div ? = 3$
(*g*) $? \div 4 = 2$
(*h*) $? \times 2 = 8$

The same combinations may then be presented with x substituted for the interrogation point, and the value of x will, doubtless, be readily obtained. Such equations as $3x = 6$, $2x = 8$, $\frac{x}{4} = 2$, and $\frac{9}{x} = 3$, should be presented without unnecessary explanations, to be solved at sight.

Other sight exercises should consist of equations similar to those given below, which may be followed by written ones in solving problems, that give rise to equations of like types:

(*a*) $x + 3x = 160$
(*b*) $x + 5x = 120$
(*c*) $x + 7x = 240$
(*d*) $x + 8x = 450$
(*e*) $6x - x = 100$
(*f*) $3x - 2x = 180$

Later sight exercises may gradually extend the equations to forms in which each member contains several terms; such as:

(*a*) $x + 3x - 2x = 100 - 80$
(*b*) $5x - 2x - x = 30 + 50$

By degrees the necessary technical words may be used by the teacher; such as, *equation*, *first member*, *second member*, *coefficient*, etc. There is no advantage in the early introduction of the axioms or of a long-drawn-out explanation to the effect that in finding the value of x to be 5 when $2x = 10$, both sides of the equation have been divided by the same number.

When transposition is introduced by such an equation as $x + 7 = 10$, there is no necessity for the picture of a balance, to show the need of a withdrawal of 7 pounds from the right scale pan when a similar weight is withdrawn from the pan on the left. The objection to the word "transposition" and to the rewriting of 7 in the second member with its sign changed, $x = 10 - 7$, may be mathematically sound, but it has no foundation in common sense, since all mathematicians solve their problems in this way without giving thought to the axiom that may be involved.

Unnecessary and superfluous explanations that befog the youthful student have no place at this time, the object being to enable the pupils to solve problems rather than to employ mathematical phraseology in the most approved mathematical fashion.

The same note of warning is perhaps necessary in the treatment of "clearing of fractions," some well-meaning people considering it advisable first to express both members in the fractional form, etc., etc. By their method, the equation

$$x + \frac{2x}{3} + \frac{3x}{4} = 4\tfrac{5}{6} \quad (a)$$

should be first written thus:

$$\frac{12x}{12} + \frac{8x}{12} + \frac{9x}{12} = \frac{58}{12} \quad (b)$$

before taking the usual shape

$$12x + 8x + 9x = 58 \quad (c)$$

although the derivation of (*c*) from (*a*) can be explained as the product of both members of (*a*) by 12, the common denominator of the fractions.

When equations containing algebraic fractions are regularly taken up, the first exercises will be, perhaps, in this form:

$$(a)\ \frac{x}{2} = 16 \qquad (b)\ \frac{x}{3} = 20 \qquad (c)\ \frac{x}{4} = 21$$

from which the pupils may be led to state that both members of (*a*) are multiplied by 2, of (*b*) by 3, and of (*c*) by 4, to produce:

$$(d)\ x = 32 \qquad (e)\ x = 60 \qquad (f)\ x = 84.$$

Exercises in the following form may constitute the next step. These should be worked on the blackboard; not that such a procedure is necessary to their solution, but to point out the steps that are usually taken in more complex equations:

$$(g)\ \frac{2x}{3} = 20 \qquad (h)\ \frac{3x}{4} = 21 \qquad (i)\ \frac{2x}{5} = 12$$

While these may be changed by division to the simpler ones

$$(j)\quad \frac{x}{3} = 10 \qquad (k)\quad \frac{x}{4} = 7 \qquad (l)\quad \frac{x}{5} = 6$$

from which the results are obtained by multiplication, the pupils should follow the customary course of beginning with the multiplication, thus getting rid at once of the fractions. This produces the following equations:

$$(m)\quad 2x = 60 \qquad (n)\quad 3x = 84 \qquad (o)\quad 2x = 60$$

from each of which the pupils obtain the value of x, as before.

Throughout all of their work, the pupils should be required to test their results by substituting in the *original* equation the value found for x.

The next type might provide more than one term in the first member; as, for instance:

$$(a)\quad \frac{x}{2} + \frac{x}{4} = 18 \qquad (b)\quad \frac{x}{2} + \frac{x}{4} + \frac{x}{8} = 21$$

$$(c)\quad x + \frac{x}{3} = 20 \qquad (d)\quad x + \frac{x}{2} + \frac{x}{3} = 22$$

While the terms of the first member can be combined at sight into a single term, the early procedure should be to multiply both members of each equation by the common denominator of the fractions. The object is, at present, to teach the general method, not the shortest possible one.

Multiplying both members of (*a*) by 4, of (*b*) by 8, of (*c*) by 3, and of (*d*) by 6, the following equations are obtained:

$$(e)\quad 2x + x = 72 \qquad (f)\quad 4x + 2x + x = 168$$

$$(g)\quad 3x + x = 60 \qquad (h)\quad 6x + 3x + 2x = 132$$

Each of these is simplified by combining the terms of the first member, as follows:

$$(i)\quad 3x = 72 \qquad (j)\quad 7x = 168 \qquad (k)\quad 4x = 60 \qquad (l)\quad 11x = 132$$

from which the answers are found:

$$(m)\quad x = 24 \qquad (n)\quad x = 24 \qquad (o)\quad x = 15 \qquad (p)\quad x = 12$$

In making the tests the value of the first member of each original equation is found by substituting for x its value, thus:

(*a*) 1st member: $\frac{24}{2} + \frac{24}{4} = 12 + 6 = 18$
(*b*) " " $\frac{24}{2} + \frac{24}{4} + \frac{24}{8} = 12 + 6 + 3 = 21$
(*c*) " " $15 + \frac{15}{3} = 15 + 5 = 20$
(*d*) " " $12 + \frac{12}{2} + \frac{12}{3} = 12 + 6 + 4 = 22$

Since the final value of the first member as thus obtained is the same in each case as the second member, the value found for x in each equation is presumably correct.

The foregoing equations are extended to include some that contain negative terms, and also to others that involve transposition. The following is one type:

$$(a)\quad \frac{2x}{3} - \frac{3x}{5} + 6 = x + \frac{3x}{10} - 31$$

Some teachers prefer to begin with transposition, thus:

$$(b)\quad \frac{2x}{3} - \frac{3x}{5} - x - \frac{3x}{10}\,[= -31 - 6] = -37$$

combining the terms of the second member into a single term.

Clearing of fractions, by multiplying both members of (*b*) by 30, the following equation is obtained:

$$(c)\quad 20x - 18x - 30x - 9x = -1110$$

Combining, the result is

$$(d)\quad -37x = -1110$$

from which the value of x may be written

$$(e)\quad x = 30$$

The extra step of rewriting (*d*) with the sign changed, $37x = 1110$, may be omitted when the pupils find it unnecessary.

The objection to the use of superfluous figures in arithmetical combinations does not apply to early work in equations. In this new subject the tendency should be at first to write too much without overdoing it, rather than too little. In (*b*), for instance, the value of the second member is written on the same line instead of rewriting the equation in this form:

$$(b^1)\quad \frac{2x}{3} - \frac{3x}{5} - x - \frac{3x}{10} = -37$$

as some mathematicians require, to avoid the apparent use of three members. As the second is superseded by the new one, it may be enclosed in brackets as is shown in (*b*).

The test in this example takes the following form, since each member is compound:

1st member: $\frac{2x}{3} - \frac{3x}{5} + 6$

$$\begin{aligned} &= \frac{60}{3} - \frac{90}{5} + 6 \\ &= 20 - 18 + 6 \\ &= 8 \end{aligned}$$

2d member: $x + \frac{3x}{10} - 31$

$$\begin{aligned} &= 30 + \frac{90}{10} - 31 \\ &= 30 + 9 - 31 \\ &= 8 \end{aligned}$$

The result in both members being the same, the value, 30, satisfies the equation; and is, therefore, correct.

In practice, the test may be shortened thus:

1st: $20 - 18 + 6 = 8$ 2d: $30 + 9 - 31 = 8$

the respective values of $\frac{2x}{3}$, $\frac{3x}{5}$, etc., being written directly.

The equations employed in these exercises should not be too complicated. As practice in solving them is only a preliminary to the solution of problems, they should not be very different from those that are produced by the latter.

The following types sometimes give trouble to the beginner:

(*a*) $x - (\frac{x}{2} + \frac{x}{3}) = 3$

(*b*) $16 - \frac{2x - 5}{3} = x + 6$

(*c*) $2x - 4\,(2x - 5) = 2$

Some pupils find it simpler to make the parenthesis positive by transposition, (*a*) becoming

(*d*) $x\,[\, = 3 + (\frac{x}{2} + \frac{x}{3})] = 3 + \frac{x}{2} + \frac{x}{3}$

and then by a second transposition becoming

(*e*) $x - \frac{x}{2} - \frac{x}{3} = 3$

This is sometimes useful in explaining the change of sign following the removal of the parenthesis.

The same method with (*b*) produces

$$(f)\quad 16 = \frac{2x - 5}{3} + x + 6$$

The second transposition in this case might be

$$(g)\quad 16 - 6 = \frac{2x - 5}{3} + x$$

When pupils are expert enough, they may write the foregoing at once in this form:

$$(h)\quad [16 - 6 =]\,\frac{2x - 5}{3} + x = 10$$

canceling the first member by a bracket and writing the result as the new second member.

In clearing of fractions, the portion enclosed in the bracket is neglected:

$$(i)\quad 2x - 5 + 3x = 30$$

In (*c*) the transposition produces the following equation:

$$(j)\quad 2x = 2 + 4\,(2x - 5)$$

from which is obtained

$$(k)\quad 2x = 2 + 8x - 20$$

In the transposition of $-\left(\frac{x}{2} + \frac{x}{3}\right)$ in (*a*), of $-\frac{2x - 5}{3}$ in (*b*), and of $-4\,(2x - 5)$ in (*c*), the pupils must be made to understand that the sign preceding the parenthesis or the fraction is the only one that undergoes a change.

In testing (*a*), (*b*), and (*c*), the respective values are substituted in the original equations just as they are stated, the expression within each parenthesis being simplified before the removal of the latter. The same course is taken with the numerator of the fraction, the line separating the former from the denominator having the effect of a parenthesis. The procedure is as follows:

(*a*) When $x = 18$

1st member: $x - \left(\frac{x}{2} + \frac{x}{3}\right)$ 2d member: 3

$= 18 - \left(\frac{18}{2} + \frac{18}{3}\right)$

$= 18 - (9 + 6)$

$= 18 - 15$

$= 3$

(*b*) When $x = 7$

1st member: $16 - \frac{2x - 5}{3}$ 2d member: $x + 6$

$= 16 - \frac{14 - 5}{3}$ $= 7 + 6$

$= 16 - \frac{9}{3}$ $= 13$

$= 16 - 3$

$= 13$

(*c*) When $x = 3$

1st member: $2x - 4\,(2x - 5)$ 2d member: 2

$= 6 - 4\,(6 - 5)$

$= 6 - 4\,(1)$

$= 6 - 4$

$= 2$

The foregoing form is followed by some teachers, who prefer to place each successive step under the previous one instead of writing all of them in a line.

Throughout the work in equations, the teacher should employ such of the problems in the arithmetic text-book as lend themselves readily to algebraic solution. Many pupils have learned to work certain types without understanding the method employed, further than to know that it obtains the answer. The fact that a general method is applicable to many varieties, throws a new light upon their previous work, which is a revelation to many pupils.

After the equation has been taught, its use should not be required of pupils who can get work satisfactorily without its aid. In the later problems, pupils should be permitted to employ any method they prefer, arithmetical or algebraic.

The proper use of the equation should finally enable the bright pupil to get along in many cases without it. He should also be able to abbreviate his solutions by the omission of steps required by his slower mates, and in this he should be encouraged.

In the later lessons the sight solution of simple equations should constitute a feature of the work.

Much time is wasted by pupils who are required to write out their work at too great length. The "model solution" in the text-book is intended to explain a method and not to furnish a form to be followed.

PROBLEM

A and B together receive \$700, B's share being \$40 more than twice A's. What is the share of each?

MODEL SOLUTION

Let x = number of dollars in A's share.
Then $2x + 40$ = number of dollars in B's share.
And $3x + 40$ = number of dollars in shares of both.
By the terms of the problem, $3x + 40 = 700$.
Transposing, $3x = 700 - 40$.
Combining, $3x = 660$.
Dividing both members by 3, $x = 220$.
etc. etc.

A pupil's work is more likely to be expressed thus:

$$\begin{aligned} x &= \text{A's} \\ 2x + 40 &= \text{B's} \\ 3x + 40 &= 700 \\ 3x &= 660 \\ \begin{array}{r} x \\ 2x + 40 \end{array} & \begin{array}{l} = 220;\ \text{A's share} \\ = 480;\ \text{B's} \quad \text{“} \end{array} \left.\begin{array}{r} \$220 \\ \underline{\$480} \end{array}\right\}\text{Ans.} \\ & \qquad\qquad \text{Test:} \quad \$700 \end{aligned}$$

While the latter form seems incomplete, it contains all the pupil needs, and he is weakened by being required to do more than is necessary, provided, however, that he is not permitted to make incorrect statements.

A detail that should not be overlooked by the pupil is the correctness of the method of testing his answer to a problem. When B's share in the above example is obtained by finding the value of $2x + 40$ when $x = 220$, the test consists in ascertaining that the sum of the shares is \$700. If a pupil obtains the value of B's share as \$480 by deducting A's share from \$700, as some are likely to do, he should test the correctness of \$480 by ascertaining its equality with the sum of \$40 and twice \$220.

It is occasionally necessary to show pupils how to abbreviate some of their "statements." The following is a typical problem which lends itself to a shortened form:

"After spending $\frac{1}{2}$ of his money, then $\frac{1}{3}$ of the remainder, then $\frac{1}{4}$ of this remainder, and finally $\frac{1}{5}$ of the sum then remaining, a man has $50 left. How much had he at first?"

Unless told that there is another way, a pupil may pursue a course somewhat as follows:

$$\begin{aligned}
\text{Let } x &= \text{original sum}\\
\frac{x}{2} &= \text{1st expenditure}\\
x - \frac{x}{2} &= \text{1st remainder} = \frac{x}{2}\\
\tfrac{1}{3} \text{ of } \frac{x}{2} &= \text{2d expenditure} = \frac{x}{6}\\
\frac{x}{2} - \frac{x}{6} &= \text{2d remainder} = \frac{x}{3}\\
&\text{etc.,}\quad \text{etc.}
\end{aligned}$$

When each expenditure is a fraction of the preceding remainder, the better process is the following:

$$\begin{aligned}
\text{1st remainder} &= \frac{x}{2}\\
\text{2d}\quad \text{"}\quad &= \tfrac{2}{3} \text{ of } \frac{x}{2}\\
\text{3d}\quad \text{"}\quad &= \tfrac{3}{4} \text{ of } \tfrac{2}{3} \text{ of } \frac{x}{2}\\
\text{4th}\quad \text{"}\quad &= \tfrac{4}{5} \text{ of } \tfrac{3}{4} \text{ of } \tfrac{2}{3} \text{ of } \frac{x}{2}
\end{aligned}$$

The pupil omits the first three steps, and knowing that the order of the factors is immaterial, writes only the final remainder, as follows:

$\frac{x}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}$, and simplifies it to $\frac{x}{5}$, the equation being $\frac{x}{5} = 50$.

The same type obtains in the following problem in trade discount:

"A man paid $342 for a piano. Find the list price when discounts of 25 and 20% were allowed, and 5% for cash."

When x represents the list price, the statement would be

(*a*) 75% of 80% of 95% of $x = 342$
or (*b*) $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{19}{20}$ of $x = 342$, replacing per cents by fractions;
or (*c*) $\frac{57x}{100} = 342$, when first member is simplified.

As a rule, an algebraic equation does not appear in the form shown at (*a*) or (*b*), the first member being changed to the form shown in (*c*), before the second member is written.

The equations given on p. 295 for the solution of problems in interest take a different shape when these problems constitute a portion of the algebraic work.

Thus, $P \times \frac{6}{100} \times \frac{5}{2} = 52.50$
expressd algebraically, appears as $\frac{3x}{20} = 52\frac{1}{2}$
$350 \times \frac{6}{100} \times T = 52.50$
expressed algebraically, appears as $21x = 52\frac{1}{2}$
$350 \times \frac{R}{100} \times \frac{5}{2}$
expressed algebraically, appears as $\frac{35x}{4} = 52\frac{1}{2}$

the required number being denoted by x, and the decimal being replaced by a common fraction.

It is frequently necessary to put pupils on their guard against the tendency to let x equal the required number. The inadvisability of the practice may be shown by means of a few problems similar to the following:

"1. The perimeter of a rectangular field measures 140 rods, and one side is 10 rods longer than the other. Find (*a*) the dimensions, (*b*) the diagonal, and (*c*) the area."

"2. How many acres are contained in a rectangular field whose breadth is 70 rods shorter than its length and whose perimeter is 460 rods?"

"3. How much does a man save in walking diagonally across a rectangular field instead of walking along two adjacent sides which together measure 170 yards, one being 70 yards longer than the other?"

After finding the dimensions (*a*) in 1, by means of an equation, the pupil will obtain (*b*) and (*c*) by arithmetical calculations. If 2 follows immediately after, and he attempts to make x represent the area, he will soon come to a halt, and ought naturally to turn to 1 for relief, or be directed thereto by the teacher. By the time he attacks the third he should begin to realize that the solution of all three problems depends upon the employment of x to represent one of the two dimensions; and that when the problem calls for other things depending upon the dimensions, the latter must first be ascertained.

A few pupils fail to observe that often a problem is partly arithmetical and partly algebraic, and that the equation need not be resorted to until some of the numbers employed therein are obtained arithmetically.

As an illustration of a simple type, a problem in profit and loss will serve:

"What per cent is gained on an article bought for \$60 and sold for \$84?"

The algebraic portion of this consists in finding what per cent of the cost, \$60, constitutes the profit, the latter, \$24, being first found arithmetically. While the problem as it stands could be stated in this equation:

$$60 + \frac{60x}{100} = 84$$

it is better that the employment of the equation be limited to such portions of a problem as are simplified by the use of x, and to discontinue the use of the latter when it becomes possible to obtain results without it.

In the problem that requires the cost of an article sold for \$84, on which the profit is 40%, the equation form is more necessary:

$x + \frac{40x}{100} = 84$, or $x + \frac{2x}{5} = 84$, although many pupils write it at once either as $\frac{7x}{5} = 84$, or as $\frac{140x}{100} = 84$

When the elementary course provides for the solution of simple problems involving equations containing two unknown numbers, only one method of elimination should be taught, that by addition (and subtraction). In the simultaneous equations:

$$(a)\ 3x + 4y = 32 \text{ and}$$
$$(b)\ 2x - 7y = 2$$

x is eliminated by multiplying (a) by 2 and (b) by 3, producing the following:

$(c)\ 6x + 8y = 64$

$(d)\ 6x - 21y = 6$

To eliminate x, (d) is subtracted from (c). Many teachers object to any change of sign of the terms of the subtrahend before performing the subtraction. It seems, however, to help beginners to permit them to write the changed sign above each term. When the value of x is found by substituting 2 for y in (*a*), as follows: $3x + 8 = 32$, $3x = 24$, $x = 8$, the results should be tested by substituting both values in (*b*), and vice versa.

$$\begin{array}{rcrcr} 6x &+& 8y &=& 64 \\ - & & + & & - \\ 6x &-& 21y &=& 6 \\ \hline & & 29y &=& 58 \\ & & y &=& 2 \end{array}$$

When $x = 8$, $y = 2$

$$\begin{array}{llll} (b) & \text{1st member:} & 2x - 7y & \qquad \text{2d member: } 2 \\ & & = 16 - 14 & \\ & & = 2 & \end{array}$$

Many of the problems solved by the use of x alone in the earlier lessons may be given with other numbers, to be worked by means of x and y.

To round out the equation work of the grammar schools, a brief treatment of quadratics may be given. The authorities are as yet slow to concede the advisability of advancing beyond the solution of linear forms containing two unknown numbers, presumably on the ground that the higher equations are beyond the capacity of pupils below the ninth year.

If the preceding algebraic topics have not been unduly extended, time can be found for the new work, which should begin with a few examples in finding the products of two binomials. The following are types:

(*a*) $(x + 3)\ (x + 3)$ (*b*) $(x - 4)\ (x - 4)$ (*c*) $(x + 5)\ (x - 5)$
(*d*) $(x + 3)\ (x + 4)$ (*e*) $(x - 5)\ (x - 6)$ (*f*) $(x + 7)\ (x - 8)$

In connection with (*a*) the pupils may be required to give orally the squares of arithmetical numbers, obtained as follows:

(*g*) $21^2 = (20 + 1)^2 = 20^2 + 2 \times 20 \times 1 + 1^2$, etc.
(*h*) $34^2 = (30 + 4)^2 = 30^2 + 2 \times 30 \times 4 + 4^2$, etc.

With (*b*) the following may be used:

(*i*) $49^2 = (50 - 1)^2 = 50^2 - 2 \times 50 \times 1 + 1^2$, etc.
(*j*) $58^2 = (60 - 2)^2 = 60^2 - 2 \times 60 \times 2 + 2^2$, etc.

With (*c*) the following sets may be used:

(*k*) $63 \times 57 = (60 + 3)\ (60 - 3) = 60^2 - 3^2$, etc.
(*l*) $84 \times 76 = (80 + 4)\ (80 - 4) = 80^2 - 4^2$, etc.

One use of finding the difference between the squares of two numbers by taking the product of their sum by their difference may be shown in an example requiring the finding of the third side of a right triangle when the hypotenuse and another side are given. When, for instance, the hypotenuse is 17 and one side is 15, the difference of their squares is equal to $(17 + 15) \times (17 - 15)$; that is, 32×2, or 64. The square root of 64 gives the length of the other side as 8. The number 64 is in this case obtained more quickly by the foregoing method than by deducting 15×15 from 17×17.

Other oral examples are:

(*a*) Hypotenuse, 13; base, 12
(*b*) " 25; " 24
(*c*) " 40; " 41

The first problems should yield equations containing in the statement x^2, which disappears as the work progresses. The following is one type:

"The difference between the squares of two consecutive numbers is 43. What are the numbers?"

Representing the numbers by x and $x + 1$, the successive equations are:

$$\begin{aligned} (x + 1)^2 - x^2 &= 43 \\ x^2 + 2x + 1 - x^2 &= 43 \\ 2x + 1 &= 43, \text{ etc.} \end{aligned}$$

If proportion has been studied, the following is a type:

"What number should be added to 1, 3, 4, and 7, respectively, in order that their sums may form a proportion?"

This may be expressed thus:

$$x + 1 : x + 3 :: x + 4 : x + 7$$

which produces the following equation, the product of the extremes being made equal to the product of the means:

$$(x + 1)\ (x + 7) = (x + 3)\ (x + 4)$$

When the indicated multiplications are performed, this becomes

$$x^2 + 8x + 7 = x^2 + 7x + 12$$

etc. etc.

The same type may be presented in this way:

"By what number must each term of $\frac{1}{3}$ and of $\frac{4}{7}$ be increased in order that the resulting fractions may be equal to each other?"

This gives rise to the equation:

$$(a)\quad \frac{1 + x}{3 + x} = \frac{4 + x}{7 + x}$$

The pupils having already been taught that equations of this kind are "cleared of fractions" by cross-multiplication, rewrite (a) thus:

$$(b)\quad (1 + x)\,(7 + x) = (4 + x)\,(3 + x)$$

and then perform the indicated multiplications to produce

$$(c)\quad 7 + 8x + x^2 = 12 + 7x + x^2$$

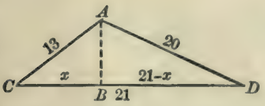
After the pupils have studied right triangles the following problem is simple:

"The hypotenuse of a right triangle is 9 yards longer than the base, and the perpendicular is 15 yards. How long is the hypotenuse?"

Representing the base by x, the hypotenuse is $x + 9$, and the equation is

$$(x + 9)^2 - x^2 = 225$$

The scalene triangle furnishes many problems, of which the following is a type:



"The base of a triangle is 21 rods, and the other sides are 13 and 20 rods, respectively. Find the altitude."

In the diagram, one segment of the base is represented by x, the other then being $21 - x$. The altitude AB divides the given triangle into two right triangles of which AB is the common perpendicular. In the first

$$\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2;$$

and in the second,

$$\overline{AB}^2 = \overline{AD}^2 - \overline{BD}^2.$$

This gives the following:

$$\overline{AD}^2 - \overline{BD}^2 = \overline{AC}^2 - \overline{BC}^2$$

Substituting for AD, BD, AC, BC, their respective values, as shown in the diagram, the equation takes this form:

$$20^2 - (21 - x)^2 = 13^2 - x^2$$

i.e. $400 - 441 + 42x - x^2 = 169 - x^2$

whence, $42x = 210$

and $x = 5$

Since $AB^2 = 13^2 - 5^2 = 144$, $AB = 12$.

As a problem of this kind does not begin with the use of x to indicate the required side, AB, the length of which is afterwards found arithmetically, its solution needs suggestions from the teacher. Such a problem is sometimes practical in calculating the area of a scalene triangle when the altitude is obtained from the sides, whose lengths are given.

These more difficult types should, however, be postponed until the end of the course, in order not to delay the pupil's acquaintance with simpler problems involving the quadratic equation.

Quadratic equations are of the two forms represented by the following:

(*a*) $3x^2 = 75$ (*b*) $x^2 + 4x = 5$

The second, (*b*), is called a *complete* quadratic equation, as it contains both x^2 and x; the first, (*a*), is called an *incomplete* one. They are also called (*a*) *pure* and (*b*) *affected*.

As a preliminary to problems involving equations of form (*a*), a few sight exercises may be given, such as

(*a*) $x^2 = 64$ (*c*) $x^2 - 14 = 50$

(*b*) $2x^2 = 50$ (*d*) $x^2 + 24 = 49$

In giving the results, the pupil need state only the positive ones, the negative values being postponed for the present.

The problem types employed are not very numerous; one is: "Find the sides of a rectangle whose area is 4800 square yards, one side being three times as long as the other."

Representing the sides by x and $3x$, respectively, the area is indicated by $3x^2$, and the equation is

$$3x^2 = 4800$$

A problem of another type is:

"The product of $\frac{2}{3}$ of a number by $\frac{4}{5}$ of itself is 480. Find the number."

The equation is:

$$\frac{8x^2}{15} = 480$$

Others are:

"What is the radius of a circle whose area is 616 square yards?"

"The volume of a square prism is 176 cubic feet and the altitude is 11 ft. Find the length of each side of the base."

An occasional problem that would ordinarily produce an equation containing x^2 and x may be solved as a pure quadratic; as, for example:

"The base of a right-triangle is 14 feet longer than the perpendicular and the hypotenuse is 26 feet. Find the area."

By making the base and the perpendicular $x + 7$ and $x - 7$, respectively, the equation is indicated thus:

$$(x + 7)^2 + (x - 7)^2 = 26^2$$

Squaring each binomial and 26, the equation becomes

$$x^2 + 14x + 49 + x^2 - 14x + 49 = 676$$

Transposing, $2x^2\ [= 676 - 98] = 578$

from which $x^2 = 289$

and $x = 17.$

The required sides are, therefore, (17 + 7) ft. and (17 − 7) ft., or 24 ft. and 10 ft., and the area is 120 sq. ft.

The test consists in ascertaining that $24^2 + 10^2$ is equal to the square of 26.

As an introduction to the study of affected quadratics, sight drills are employed in which the pupils give the squares of such binomials as

(*a*) $x + 1$ (*b*) $x - 1$ (*c*) $x + 3$ (*d*) $x - 5$ etc.

Other exercises call for the square roots of such trinomials as

(*e*) $x^2 - 14x + 49$ (*f*) $x^2 + 12x + 36$ etc.

When pupils have learned that the square of a binomial consists of three terms, the next exercise may consist in the completion of the square of the following binomials by stating the absolute term:

(*g*) $x^2 + 4x$ (*h*) $x^2 - 10x$ (*i*) $x^2 + 20x$ etc.

A pupil writes on the blackboard from the teacher's dictation a number of examples of these types as follows:

$$(a)\ x^2 + 4x = 12 \qquad (b)\ x^2 - 10x = -24 \qquad \text{etc.}$$

A pupil at his seat gives (*a*), another gives (*b*) in the following forms, and so on, the pupil at the blackboard writing each derived equation under the original:

$$(c)\ x^2 + 4x + 4 = 16 \qquad (d)\ x^2 - 10x + 25 = 1 \qquad \text{etc.}$$

Other pupils give in succession the square roots, which are written underneath, as follows:

$$(e)\ x + 2 = \pm 4 \qquad (f)\ x - 5 = \pm 1$$

The next step gives the values of x, which are announced by a different pupil for each equation, and are also written by the pupil at the blackboard:

$$(g)\ x = 2 \text{ or } -6 \qquad (h)\ x = 6 \text{ or } 4$$

This exercise is concluded by the statement of the tests:

(*i*) When $x = 2$
the 1st member of (*a*)
becomes $4 + 8$, or 12.
When $x = -6$
it becomes
$36 - 24$, or 12.

(*j*) When $x = 6$
the 1st member of (*b*)
becomes $36 - 60$, or -24.
When $x = 4$
it becomes
$16 - 40$, or -24.

With these examples before them on the blackboard, several of the same type may be solved as written work, the forms given in (*c*) and (*d*) being possibly extended in this way:

$$(c)\ x^2 + 4x + 4\ [= 12 + 4] = 16$$
$$(d)\ x^2 - 10x + 25\ [= -24 + 25] = 1$$

to show more clearly that both members have been increased by the same number, in (*c*) by 4 and in (*d*) by 25.

As the majority of the equations are likely to have integral roots, the teacher is frequently tempted to employ the factor method of solution. At this stage, however, it is inadvisable to introduce more than one way, and the method of completing the square, etc., being applicable to all quadratic equations, is the best for the elementary pupil. The object should be to give an acquaintance with quadratic equations rather than to develop ability to work rapidly.

When the elementary pupil reaches the equation $x^2 - 3x = 4$, for instance, he should complete the square in the regular way, writing the derived equations as follows:

$$\begin{aligned} x^2 - 3x + \left(\frac{3}{2}\right)^2 &[= 4 + \frac{9}{4}] = \frac{25}{4} \\ x - \frac{3}{2} &= \pm \frac{5}{2} \\ x &= \frac{5}{2} + \frac{3}{2} \text{ or } -\frac{5}{2} + \frac{3}{2} \\ &= \frac{8}{2} \text{ or } -\frac{2}{2} \\ &= 4 \text{ or } -1 \quad \textit{Ans.} \end{aligned}$$

Both members of an equation in the form

$$6x^2 - 13x = 28$$

should be divided by the coefficient of x^2, giving

$$x^2 - \frac{13x}{6} = \frac{28}{6}$$

and the square completed, thus:

$$x^2 - \frac{13x}{6} + \left(\frac{13}{12}\right)^2 = \frac{28}{6} + \frac{169}{144} = \frac{672 + 169}{144} = \frac{841}{144}$$

etc. The various devices to get rid of fractions tend merely to mystify the beginner, as would the attempt to factor the first member of

$$6x^2 - 13x - 28 = 0$$

Practical problems producing quadratic equations are not very numerous. One type has been solved already, and consists in finding, for instance, the remaining sides of a triangle when their difference is given as 7 inches and the hypotenuse as 17 inches, the equation being

$$x^2 + (x + 7)^2 = 17^2$$

Another consists in finding the side of the base of a square prism when the entire surface is given as 190 square feet and the altitude as 7 feet. When x represents a side of the base, the area of the two bases will be $2x^2$, and that of the four sides constituting the convex surface will be $28x$. The equation becomes

$$2x^2 + 28x = 190$$

which gives,
$$x^2 + 14x = 95$$

Other problems may be selected from text-books.

In giving answers to problems, the negative root of a quadratic equation should be rejected as superfluous.

Some teachers prefer the following method of obtaining mentally the square of an arithmetical number:

$$\begin{aligned}21^2 &\equiv 22 \times 20 + 1^2 \equiv 440 + 1 \equiv 441\\ 34^2 &\equiv 38 \times 30 + 4^2 \equiv 1140 + 16 \equiv 1156\\ 49^2 &\equiv 50 \times 48 + 1^2 \equiv 2400 + 1 \equiv 2401\\ 58^2 &\equiv 60 \times 56 + 2^2 \equiv 3360 + 4 \equiv 3364\end{aligned}$$

The underlying principle is shown by the following equations, the first of which represents the difference of the squares of 21 and 1 as the product of the sum of 21 and 1 by their difference, $21 - 1$.

$$21^2 - 1^2 \equiv (21 + 1)\ (21 - 1) \equiv 22 \times 20 \quad (a)$$

Transposing -1^2, $21^2 \equiv 22 \times 20 + 1^2$

The method consists in employing as one of the two factors of the second member of (a) a number consisting of tens alone. This is done in the foregoing illustration by decreasing 21 by 1. The other factor is, therefore, 21 increased by 1. To the product of these factors the square of 1 is added.

The same result is obtained by making 30 $(21 + 9)$ one of the factors, the other being 12 $(21 - 9)$, their product being increased by 9^2.

$$\begin{aligned}21^2 &\equiv 30 \times 12 + 81 \equiv 360 + 81 \equiv 441\\ 34^2 &\equiv 40 \times 28 + 36 \equiv 1120 + 36 \equiv 1156\\ 49^2 &\equiv 40 \times 58 + 81 \equiv 2320 + 81 \equiv 2401\\ 58^2 &\equiv 50 \times 66 + 64 \equiv 3300 + 64 \equiv 3364\end{aligned}$$

In order to show the use of the sign of identity ($\equiv$), it is employed above. In the formal study of algebra, the use of the sign of equality ($=$) is restricted to equations, from which the value of an unknown number is to be ascertained; as

$$(x + 5)\ (x - 5) = (x - 1)\ (x - 3)$$

which is read, "The product of $(x + 5)$ by $(x - 5)$ is equal to the product of $(x - 1)$ by $(x - 3)$."

To indicate that the product of $(x + 5)$ by $(x - 5)$ is $x^2 - 5$, it is expressed thus:

$$(x + 5)\ (x - 5) \equiv x^2 - 5$$

which is read, "$(x + 5)$ multiplied by $(x - 5)$ *is* $x^2 - 5$, or *is the same as* $x^2 - 5$, or *is identical with* $x^2 - 5$."

CHAPTER II

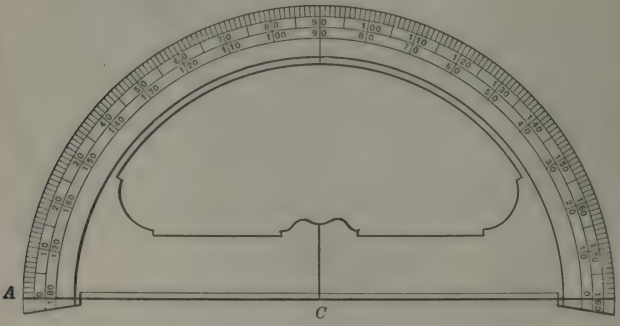
CONSTRUCTION EXERCISES

Some courses of study that do not provide for the teaching of equations in the elementary school favor practice in such drawing exercises as tend to give the pupils an acquaintance with important facts in elementary geometry.

In the early exercises a protractor is used to lay off lines meeting at a given angle, and a right triangle is used to erect perpendiculars and to draw parallel lines. In later exercises the use of these tools is discontinued, lines meeting at certain angles, parallels, etc., being constructed by means of compasses, no other instrument being employed except the ruler and the pencil.

A beginning may be made by the freehand drawing on the blackboard of two lines to form (*a*) one angle, (*b*) two angles, (*c*) four angles.

The next step consists in using the protractor to draw two lines meeting at a given angle.

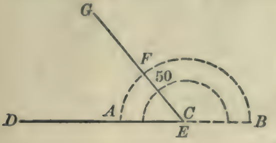


The simplest form of protractor is the semicircular one, made of brass or of cardboard. On the cardboard protractor, which can be procured in quantities for about a cent apiece, the numbers are easily distinguished.

The type here shown has for its base AB, the diameter of a semicircle, of which C is the center.

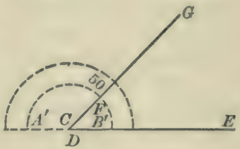
The protractor generally has two sets of numbers, one beginning with 0° at A, and extending to 180° at B, the other beginning with 0° at B' and extending to 180° at A'. To avoid confusing the pupils at the outset, the first exercises should be the drawing of a line from the right extremity of a horizontal line, making with the latter an angle of (*a*) 30°, (*b*) 50°, (*c*) 90°, (*d*) 110°, (*e*) 150°, etc.

If a blackboard protractor and compasses are available, the method may be illustrated on the blackboard.



The horizontal line DE is first drawn. The middle point C of the base of the protractor is placed on E, the extremity of the line, AC being made to coincide with AE.

Taking the outside row of figures, a dot is placed at F, 50° from A, the protractor is then removed, and the line EG is drawn through the point F.



The next series may call for the formation of angles of various sizes at the left extremity of DE. In these the inside row of figures is used.

After a little experience the average pupil should be able to approximate the size of an angle at sight, which will enable him in subsequent exercises to correct any tendency to use the wrong row of figures.

In the foregoing exercises it is customary to note in each angle its contents in degrees.

A general example calling for the construction of an angle of 50° should produce a variety of forms from the different pupils. The first line may be either horizontal, vertical, or oblique. The second line may be drawn above or below, and from either end. Pupils should understand that the avoidance of set forms constitutes a species of merit, and that two pupils may compare notes during the progress of their work in order to obtain a different arrangement.

The drawing exercises throughout the course should be accompanied by a few appropriate oral or sight exercises. The size of the angle formed by the hands of a clock at 1 o'clock, 2 o'clock, 3 o'clock, etc., may be asked at this time. In giving the angle at 7 o'clock the pupil may also be asked the number of degrees passed over by the hour hand. There is no need in the elementary school of discussing straight angles or angles of 210°; the pupil should call the 7 o'clock angle one of 150°.

If it be deemed worth while, a written exercise may call for the angles at 2 : 15, 8 : 40, etc., respectively; to each of these problems the pupil may obtain the correct answer by carefully drawing the clock face and remembering that the hour hand moves as well as the minute hand.

Another set of exercises consists in drawing a line making two angles with a given line, one of them being (*a*) 30°, (*b*) 50°, (*c*) 120°, etc. In each of the two angles is written its size in degrees.

The variation in the first line already recommended should appear in the work of different pupils, and the second line need not be drawn to the middle point of the first.

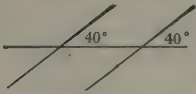
A few exercises of this kind show the pupils that the sum of two supplementary angles is 180°.

Other exercises illustrate the equality of vertical (opposite) angles by requiring the pupils to draw two lines making four angles, one of which is (*a*) 20°, (*b*) 75°, (*c*) 150°, etc., writing in each angle its contents in degrees and also measuring the number of degrees in each of the remaining three angles.

When the sum of the four angles about a given point is found to be 360°, the pupils may be asked to draw lines forming (*a*) 3, (*b*) 4, (*c*) 5, (*d*) 6 equal angles about a given point, after having given oral replies to questions as to the size of each angle in a required set.

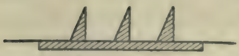
During this and other exercises the pupils may be sent to the blackboard, in turn, to draw one of the exercises prescribed, sometimes using the ruler and the protractor, and sometimes doing the work freehand. The pupil in the latter case estimates the divergence called for by the number of degrees stated by the teacher, and writes it in place. The size of each of the other angles is then ascertained by calculations based upon the knowledge obtained through the exercises already done with the help of the protractor, and when thus ascertained, each is written where it belongs.

An acquaintance with parallel lines is begun by an exercise calling for the drawing of a line intersecting a horizontal line at an angle of 40°, for instance; and the drawing of a second oblique line at a distance of an inch or so from the point of intersection, making the same angle of 40° with the horizontal line, and running in the same direction. The contents (in degrees) of the remaining six angles may next be written.



If the pupil makes a mistake in the number of degrees assigned to any angle, the reason for the mistake should be determined.

The employment of the right triangle in drawing parallel lines is another exercise. The triangle is slid along the ruler, the latter being held steadily in place. Successive lines drawn along either the perpendicular or the hypotenuse of the triangle will be parallel to each other.



Next may come the employment of the ruler and the triangle to draw a line parallel to a given line. One edge of the triangle, for instance its perpendicular, is placed along the given line, and the ruler is placed against the base. The ruler is then held steadily, while the triangle is moved along it to the point through which the parallel line is to be drawn, the drawing being done by using the appropriate side of the triangle as a ruler. If necessary, the line thus located may be lengthened.

Exercises in constructing triangles may begin by drawing from each end of a 3-inch line a line making an angle of 60° with the base. In exercises of this kind the given (3-inch) line is drawn heavily, the next line is drawn lightly and of indefinite length, the third is drawn heavily and terminates when it reaches the second. The latter is then darkened to the proper point, but the excess need not be erased. The problem generally requires the pupil to measure the length of the last two lines and to write on each its length in inches. The length of the first line is written when the line is drawn and also the contents of each of the two base angles. The task is completed by measuring the third angle with the protractor and inserting the ascertained number of degrees.

When the drawing is accurate, the two sides will measure 3 inches each and the third angle will contain 60°. If this is not found to be the case, a pupil should go over his work and ascertain the origin of the mistake.

Much of the benefit derivable from this work is obtained through care in drawing. To this end, a hard pencil with a sharp point is important.

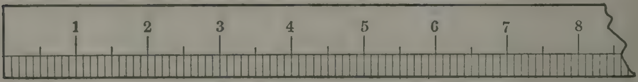
The base line may be vertical or oblique, as well as horizontal. In the latter case the triangle may be drawn below the base. Equilateral triangles may be drawn with a base line of any length, each of the two base angles containing 60°.

The next type of triangle may have a base of 3 inches and two base angles of 70° each. Upon the completion of the triangle, its sides are measured as accurately as possible, also the remaining angle, and the results inserted in the proper places. When the drawing is accurate the third angle will contain 40° and the two sides will be of the same length. Questions by the teacher should develop the fact that the sum of the three angles of the triangle is 180°, that the sides opposite equal angles in the same triangle are equal, and that the smallest angle is opposite the smallest side.

The pupils should be informed that the unequal side of an isosceles triangle is called the base; and they should be required in drawing these triangles to employ occasionally for the base an oblique line or a vertical line, and also to have the triangle fall below the base when the latter is horizontal.

After several isosceles triangles have been drawn on a given base with given base angles, one may be constructed having a side of 2½ inches, for instance, and two angles of 50° and 80°, respectively. The pupils should discover that the required triangle is isosceles, having a second side of 2½ inches and a second angle of 50°. A similar result is obtained when one angle contains 45° with a second of 90°, etc.

In order to avoid fractions in measuring lines, many teachers use a metric ruler. One type of a foot rule contains the inch divisions on one side, the other side being divided into 3 decimeters, each subdivided into centimeters and millimeters. The side marked in inches shows 96 subdivisions, 84 of them containing fractions; the metric side gives 300 subdivisons, all of which can be stated as whole numbers. Thus a line about 5 $\frac{7}{16}$ inches in length may be expressed as 138 millimeters.



As a type of scalene triangle, a base line of 2½ inches may be used and base angles of 37° and 53°, respectively. The remaining angle will contain 90° and the remaining sides will measure 1½ and 2 inches, respectively. The facts that the three angles of this triangle together contain 180°, and that the greatest angle is opposite the greatest side, etc., are again developed; also that only one triangle can be drawn with two given angles and a given included side. These facts are confirmed by the drawing of other triangles, when time warrants.

The construction of a triangle having sides of 1½ and 2 inches, respectively, forming a right angle may be employed to show that only one triangle can be constructed with two given sides and a given included angle.

Before leaving this portion of the work in triangles, an exercise in dividing a 2-inch equilateral triangle into 1-inch triangles by marking off on each side its middle point, and in drawing three lines, one connecting each two points, should be given, in order to enable the pupils to see for themselves the ratio of the area of the 1-inch triangle to that of the 2-inch triangle. A 3-inch equilateral triangle may also be subdivided into 1-inch triangles, from which the ratio between their areas is readily determined. The number of degrees in each angle of one of the small triangles should be compared with the number in each angle of the large triangle.

A right triangle having sides of 3, 4, and 5 inches, respectively, might also be subdivided into four equal triangles and their sides measured. The angles of the large triangle should also be compared with those of each of the small triangles.

High-school pupils not infrequently think that the sides of a triangle four times as large as one having sides of 3, 4, and 5 inches are 12, 16, and 20 inches, respectively, and that the angles of such a triangle would be 148°, 212°, and 360°, respectively, when the angles of the smaller one are 37°, 53°, and 90°, respectively.

The oral exercises may include questions as to the value of the equal angles in an isosceles triangle, when the angle opposite the base measures (*a*) 40°, (*b*) 60°, (*c*) 70°, (*d*) 50°, etc. Pupils may find the number of degrees in the remaining angles when one angle of an isosceles triangle measures (*a*) 90°, (*b*) 100°. The number of degrees in the remaining angles of a right triangle may be required when one angle measures (*a*) 30°, (*b*) 40°, (*c*) 50°, etc. Two angles of a scalene triangle may be given, from which the pupils are required to find the third. A question may be put as to the contents of each angle in a 6-inch equilateral triangle when the angles in a 1-inch triangle measure 60° each.

The construction of quadrilaterals may consist in drawing the various types by means of the right triangle, protractor, etc. The base line need not always be horizontal. The construction of a rhomboid by first drawing two lines of 3 inches and 2 inches, respectively, meeting at an angle of, say, 60°, and the completion of the rhomboid by employing the ruler and the triangle to draw the remaining sides, should show the equality of the opposite sides and angles and that the sum of the angles is 360°.

The construction of a parallelogram on a 3-inch base and with an altitude of 2 inches should teach the pupils that an indefinite number can be formed which will satisfy the given conditions. Parallel with the base, and 2 inches from it, a line is drawn of indefinite length. To any two points 3 inches apart, in this line, lines are drawn from the extremities of the base, thus forming a parallelogram that will be either a rectangle, a rhombus, or a rhomboid, according to the location of the two points in the upper line. The equality in area of all of these different forms should be shown.

The variety of areas that may be contained in several 3-inch rhombuses, or in rhomboids having sides of 3 inches and 2 inches, respectively, may be developed by the construction of two or more, one of each variety containing one angle of, say, 40°, and a second containing one angle of 80°. A third rhombus or rhomboid containing one angle of 100°, when compared with the other two, will show its equality with one of them, and the reason therefor should be developed. The pupils should be required to ascertain what data are necessary to indicate the shape as well as to define the area of a parallelogram. They will discover that the length of two adjacent sides and the size of the contained angle will satisfy the requirements. For purposes of arithmetical calculation, the latter angle is replaced by the altitude, whose length is based both upon the contents of the angle and the length of the second side.

In these exercises, as in the earlier ones, there is no reason why a pupil should not observe the work of his neighbors, provided his own takes an entirely different form. This will be the case if he begins with a horizontal line when another employs a vertical line, or a third starts with an oblique line. In the later problems two pupils may reach the same result by different means, and the blackboard solution should give an opportunity for suggestions from pupils who think they have discovered a better way than any previously presented.

The less help given to pupils in the solution of geometrical problems the better. The teacher's chief task should be so to arrange the exercises that the solution of one gives the hints necessary to the solution of a later one. If the step between two successive exercises is found to be too great, the teacher should introduce an exercise that will make the connection between the others more apparent, rather than show the method of solving the second one.

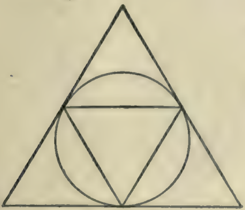
The various uses of the compasses are brought out in connection with exercises employing circles and circular arcs. If the pupils are not acquainted with arcs, chords, sectors, segments, etc., a first exercise may consist in drawing in a circle, a diameter, a radius or two, several chords, etc.

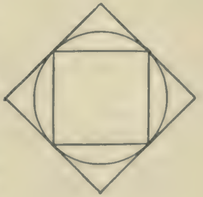
To set off on a circumference an arc of a given size the protractor is used, its center being placed on the center of the circle and two points, the required number of degrees apart, being located. It makes no difference whether these points fall without or within the circle. A ruler is placed on the center of the circle and on each point successively, and two very short lines are drawn across the circumference which denote the points that bound the arc. If a sector is required, two lines are drawn at once from the center to the circumference through the points located by means of the protractor. Exercises in constructing regular inscribed polygons begin with the location of two points bounding an arc of (*a*) 120°, (*b*) 90°, (*c*) 72°, (*d*) 60°, (*e*) 45°, etc. When the boundaries of one arc are thus indicated the distance between the points is measured by the compasses, which are used to set off the other arcs, until the circumference is divided into three arcs of 120° each, four of 90° each, etc.

An inscribed equilateral triangle, square, regular pentagon, etc., is constructed by connecting the ends of the appropriate arcs. It may be necessary in this connection to develop the fact that the distance between the compass-points denotes the length of the chord (even when this is not drawn) and not the length of the arc.

Lines, arcs, circles, etc., called for by the conditions of the problem, should be drawn in heavy lines; other lines, arcs, etc., used in locating the former, should be drawn lightly, and not unnecessarily long. The latter should not, however, be erased. Gradually the pupil will learn to omit unnecessary construction lines and to limit the length of the others. He will also learn to reduce to a minimum the inaccuracy necessarily due to the imperfect character of the tools.

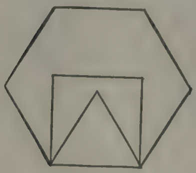
When the pupils have laid off an arc of 60° by means of the protractor, and have used the compasses to lay off the remaining arcs, the fact that the compass points are the same distance apart as in drawing the circle should be brought out, and that a protractor is not necessary in the construction of an inscribed hexagon. Advantage is taken of the same fact when it is required to inscribe an equilateral triangle, the necessary arcs of 120° consisting of two of 60° each. The relation between an inscribed equilateral triangle and a





circumscribed one is shown at once when the latter is so drawn that the angular points of the former are at the center of each side of the other. By a similar method of locating the sides of a circumscribed square, its area as compared with that of the inscribed square is shown. In drawing these circumscribed polygons, the pupils must use correct methods to derive proper benefit from the work.

Before discontinuing the use of the protractor, pupils should construct various regular polygons on a given line. This requires that the number of degrees in each angle of a regular polygon be ascertained. No formula should be given. To inscribe a regular polygon the pupils have found that an arc of 90°, 60°, 72°, etc., has been required, depending upon the number of sides in the polygon. The division of a regular hexagon into six equal equilateral triangles will show that each of its angles is made up of two adjacent angles of two of the six triangles that constitute the hexagon. As each of these angles is 60°, this gives 120° as the size of each of the six angles of the hexagon. By means of blackboard drawings, the pupils can determine the size of each of the five central angles, 72°, when a pentagon is divided by radii into five equal isosceles triangles. The size of each of the equal angles in one of the isosceles triangles is next determined to be $[(180° - 72°) \div 2]$, or 54°, the sum of two adjacent angles in two triangles, 108°, being the size of each of the five angles of a regular pentagon. While the result, 108°, is determined at once by deducting one-fifth of 360° from 180°, the pupils should employ the longer method, which is more readily understood by beginners.



An exercise that interests the average pupil consists in the construction of an equilateral triangle, a square, a regular pentagon, etc., on a given base, the pupil using his protractor to determine the direction of each of the successive lines. These may be drawn of the required length by means of the ruler, or the base may be left unmeasured, each remaining side being drawn indefinitely, and the length then cut off to the proper size by means of the compasses, the distance between the points of which is made equal to the length of the base line. As each corner is located, the center of the protractor is placed thereon and the direction of the next line indicated. The last side may be drawn without the aid of the protractor; but its length should be tested by the compasses, and the number of degrees in each of the last two angles by the protractor.

The exercises to be taught are generally prescribed in a syllabus, which specifies those in which the protractor and right triangle may be employed, and also those in which their use is forbidden. In the latter, which are considered the more valuable ones, angles and arcs are set off exclusively by the compasses.

Unless the syllabus sets forth the order to be followed, the teacher should be guided by conditions, not attempting the completion of the prescribed work in the construction of angles before constructing triangles, etc.

When there is time for preliminary work in drawing circles, some exercises are given in constructing from given data two concentric circles, two eccentric circles, two circles tangent externally, two tangent internally, etc.

The construction of triangles is led up to by the following:

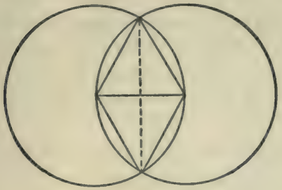
"With centers 2 inches apart, draw two circles

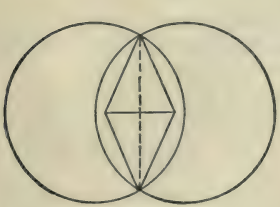
"(*a*) Each having a 2-inch radius.

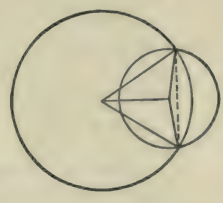
"(*b*) Each having a 2½-inch radius.

"(*c*) One having a 1½-inch radius, and the other having a 2½-inch radius.

"Connect the centers of each two circles and also each of these centers with each point of intersection of the two circles. Connect the two intersecting points by a dotted line."







From the discussion of these problems the pupils learn that the quadrilateral in each exercise is divided by the line connecting the centers into two equal triangles, those in (*a*) being equilateral, in (*b*) isosceles, and in (*c*) scalene; also that the dotted line in each indicates the double altitude of each triangle; and that in (*a*) and (*b*) it bisects the base.

A succeeding problem may be stated as follows:

"(*a*) Construct two 2-inch circles with centers 1½ inches apart. (*b*) Connect the centers. (*c*) Connect the points of intersection with the centers and with each other."

The fact is then developed that both triangles are isosceles, and that the 1½-inch base line has been bisected by the line joining the intersecting points.

Another problem is the following:

"Bisect a given line."

The pupils learn gradually that entire circles need not be drawn; also that the radii of the circles need not be measured, but that the radii of both should be equal, and longer than one-half of the given line. The necessity of two intersections is brought out, as well as the advantage of having one above and the other below the given line, as two intersections both above or both below the base would require the use of a second set of two equal radii, etc.

The intersecting arcs are lightly drawn, the two arcs of the first circle being necessarily of indefinite length. Those of the second circle, which locate the points of intersection, need only be long enough to be distinctly visible. The ruler is then placed on the two intersecting points, but only a very short line is required to denote the point of bisection.

A line to be bisected may be vertical or oblique, as well as horizontal.

The next problem may be:

"Erect a perpendicular at the middle point of a line."

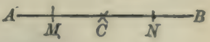
This is the preceding one in another form. The two intersecting points are located as before, and the ruler placed on both.

Beginning at the given line, a line is drawn of indefinite length in the direction of either intersecting point. It may extend either downwards or upwards, the word "erect" not necessarily implying "upwards" in this connection.

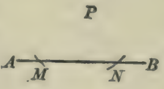
The greatest possible variety of drawings should appear among the pupils' solutions, and a blackboard drawing should be criticized by any pupil who observes a mistake, or who can suggest a possible improvement. In many of the later exercises, several methods of solving the same problem will be offered by the brighter pupils, and the teacher is occasionally surprised by one entirely new to him, and the correctness of which is not always evident at a glance.

The amount of help given by the teacher should gradually be diminished. An occasional problem may be given a day or two in advance of the time its solution is required. A description of the method employed may be written as one of the regular composition exercises.

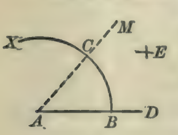
"Erect a perpendicular at a point outside of the center of a line" is another problem.



The procedure is indicated in the accompanying figure, in which *C* in the line *AB* denotes the point at which the perpendicular should be erected. This problem is made practically the same as the preceding one by using *C* as a center and marking off points *M* and *N* equidistant from *C*. This now calls for the erection of a perpendicular at *C*, the middle point of the line *MN*. As one point is settled (*C*), a single intersection, either above or below, is sufficient, *M* and *N* being taken as the centers.



"Let fall a perpendicular to a line from a point without" is another type. When *AB* is the given line, and *P* is the given point, the latter is taken as a center to locate *M* and *N* on the line. With *M* and *N* as centers, an intersection below *AB* determines with *P* the direction of the perpendicular. The latter begins at *P* and terminates at *MN*. *AB* may be vertical or oblique.



"Construct an angle of 60°" is related to the first problem, which consists of the construction of an equilateral triangle. With *A* as a center and with any convenient radius, an arc, *BX*, is drawn of indefinite length. With *B* as a center, a short arc is drawn intersecting the first arc at *C*. The line *AM* drawn through *C* makes an angle of 60° with *AD*.

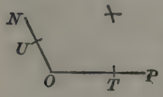
"Construct an angle of 30°" is a problem that consists practically of the bisection of an angle of 60°. The preceding figure indicates the procedure employed in constructing the latter angle. Without drawing *AM*, the points *B* and *C* are used as radii to draw two short arcs intersecting at *E*. A line drawn from *A* through *E* forms with *AD* an angle of 30°.

The pupil should understand that the chord of the arc *BC* is bisected, the problem being really reduced to the bisection of an imaginary line *BC* by a perpendicular, the radius which bisects the chord also bisecting the arc. Only one intersection is required, since *A* constitutes the other.

Any angle is bisected in this way, the points cut by an arc extending through both of the lines that form the angle, denoting the centers from which the next intersection is located.

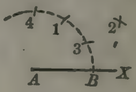
"Bisect a given angle" is next in order.

The bisection of an angle, say *NOP*, does not require that an arc be drawn through *T* and *U*, for instance; only the short portions shown in the illustration being necessary. In using these intersections as centers, the same radius should be retained to avoid unnecessary shiftings of the compass-points.

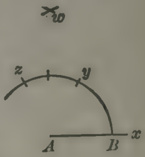


"Construct a right angle."

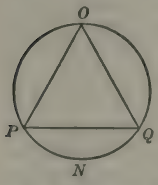
The erection of a perpendicular at *A*, one end of the line *AX*, is done in several different ways. One consists in locating *1*, making *B1* an arc of 60°; then in finding the intersection at *2*, and using it to locate *3*, 30° from *B*. Retaining the compass-points in the position used for the original radius, they will also locate *4*, when one is placed at *3*, the arc *34* being 60°. As *B3* = 30°, *B4* will be 30° + 60°, or 90°.

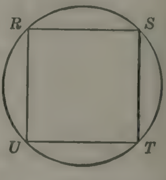


The second method lays off *By* 60° and *yz* 60°. The bisection of *yz* by means of arcs intersecting at *w* gives the direction of the perpendicular from *A* to *w*. This method requires the extension of the first arc, and is not available when the point *A* is very near the edge of the paper.

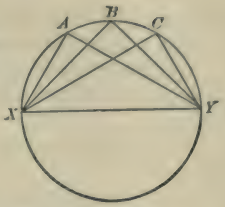


As a preliminary to a third method of constructing a right angle, the fact that an inscribed angle, that is, one having its vertex on the circumference, is measured by one-half the intercepted arc, may be developed by means of an inscribed equilateral triangle or square. In the former the sides *OP* and *OQ* of the angle at *O* intercept the arc *PNQ*, which measures 120° (⅓ of 360°). As the angle at *O* is known to contain 60°, which is one-half of *PNQ*, the pupils appreciate the application to this case at least. In the square *RSTU* the sides *SR* and *ST* intercept the arc *RUT*, 180°, and as the angle at *S* contains 90°, the rule again applies. Any

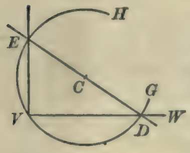




inscribed angle, therefore, whose sides intercept an arc of 180° is a right angle. Two lines drawn from any point in a circumference, A, B, C, etc., to the extremities of the diameter form a right angle.



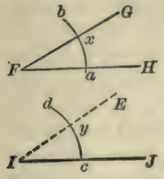
To erect a perpendicular at V, any convenient point C is taken. With C as a center and CV as a radius an arc GH is drawn, cutting VW at D. Through DC a diameter DE is drawn. A line from V passing through E is perpendicular to VW, since its sides VD and VE intercept an arc of 180°.



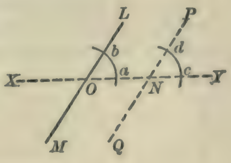
Problems in the construction of angles should at this stage be limited to those of (*a*) 60°, (*b*) 120°, (*c*) 30°, the combinations (*d*) 90° (60° + 30°), (*e*) 150° (120° + 30°), and the halves of some of the latter, for instance, (*f*) 45° (½ of 90°) and (*g*) 75° (½ of 150°); also (*h*) 135° (60° + 75°).

"Construct an angle equal to a given angle" is a problem frequently used.

When GFH is the given angle and it is required to draw from I a line making with IJ an angle equal to GFH, an arc ab is drawn with F as the center; then with I as a center and using the same radius, an arc cd is drawn. The length of the chord of ax is measured by the compasses, and an equal distance, cy, is laid off on the arc cd. A line drawn through Iy solves the problem. The arc ab may be replaced by intersections at a and x; but the arc is generally drawn to point out its relation to the arc cd.



Another type is "Draw a line through a given point parallel to a given line." To draw a line parallel to LM, for instance, through the point N, XY is drawn passing through N and cutting LM at O. At N an arc cd is laid off equal to the arc ab, which measures the angle LOY. The line PQ drawn through d and N is parallel to LM, since both lines make equal angles with XY.



The customary problems in the construction of triangles are based upon previous exercises. They may be graded somewhat as follows:

"On a 2-inch base construct:

"(*a*) An equilateral triangle.

"(*b*) An isosceles triangle having equal sides measuring 1½ inches each.

"(*c*) A triangle whose other two sides measure, respectively, 1½ and 2½ inches.

"(*d*) An isosceles right triangle.

"(*e*) A right triangle having a base angle of 60°.

"(*f*) A triangle having base angles of 60° and 30°, respectively.

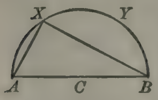
"(*g*) A right triangle having an altitude of 2 inches.

"(*h*) An isosceles triangle having an altitude of 2 inches.

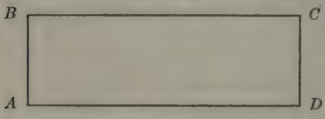
"(*i*) A scalene triangle having an altitude of 2 inches."

From these and similar problems the pupils will learn that only one triangle can be drawn when there are given, either (*a*) the three sides, (*b*) two angles and the included side, (*c*) two sides and the included angle. Any variation in the appearance of the papers should be due merely to the direction given the base line. On the other hand they will observe that innumerable shapes may be given to triangles having the base and the altitude given, although all of them will be of the same area.

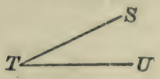
A right triangle may be constructed when any two sides are given. When one of the latter is the hypotenuse, a semi-circumference is described on it as a diameter, and from one extremity a chord is drawn corresponding in length to that of the given oblique side. Thus, a right triangle having a hypotenuse of 4 inches and an oblique side of 2 inches is constructed by drawing a semi-circumference having a radius of 2 inches, and the diameter *OAB*. With either *A* or *B* as a center, and with a radius of 2 inches, an arc is drawn, cutting the semi-circumference at *X* (or *Y*). The triangle *AXB* (or *AYB*) satisfies the conditions.



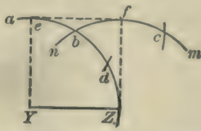
Problems in constructing quadrilaterals are modifications of those heretofore employed in exercises with triangles, as quadrilaterals consist of two triangles having a common base.



When the sides of a rectangle are given, the first step requires the erection of a perpendicular, *AB*, at one end of the line, *AD*, using the given lengths. The perpendicular *CD* may then be erected and *CB* drawn. The result is tested by measuring BC, which should equal AD, and by measuring the angles at C and B. It may also be constructed in other ways.

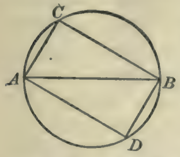


After two adjacent sides of any parallelogram are drawn, such as TU and TS, S and U may be used as centers to locate the fourth corner, the radius used in the first case being equal to TU and that in the second case to TS.



A square is drawn on YZ by using arcs each having a radius equal to YZ. The first arc is Za, with Y as a center. With Z as a center, nm is drawn, the intersection at b making Zb 60°. With center at b, c is located; then the ruler is placed on Y and c to locate d. With d as a center, e is located, 90° from Z. With e as a center, f is located; Ye, ef, and Zf then constitute the remaining sides of the square.

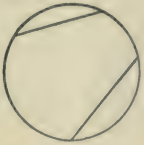
When one side of a rectangle is given and its diagonal, the problem of completing the rectangle consists in first constructing a right triangle, having the hypotenuse and another side given, and then duplicating the triangle. This is done either in the manner just described, or by drawing the entire circumference, laying off the required arcs at C and D, etc.



A hexagon is constructed upon a given line by locating the apex of an equilateral triangle of which the given line forms one side. Taking this apex as the center, a circle is drawn having a radius equal in length to the given line. The latter then becomes the chord of an arc of 60°, and other equal chords are laid off for the remaining five sides of the hexagon.

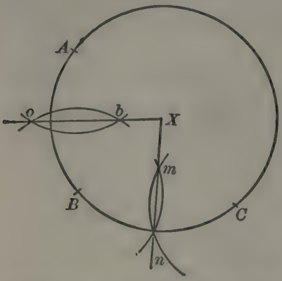
To construct an octagon on a given line, a line of the same length is drawn from each end of the former, making with it an angle of 135°. This is continued until the octagon is completed.

The construction of a circle that will pass through three given points not in the same straight line is developed through several problems. One consists in drawing a circumference by means of a tumbler or some other round object, and requiring that its center be found. This is done by drawing two chords not parallel, and bisecting each by a perpendicular. The intersection of the two perpendiculars will be the required center.



Another type may require that a circle be circumscribed about a given triangle. Two of the sides of the latter are bisected and the circle drawn. The bisection of the third side is unnecessary.

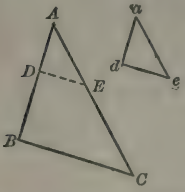
The passing of a circle through the three points not in a line may be done by joining the points to form two sides of a triangle; but this is unnecessary. To draw a circle through A, B, and C, for instance, an arc having B as a center intersects one having C as a center at m and n, and a line drawn through these points passes through the center of the required circle. Similar arcs with centers at A and B, respectively, give intersections at b and o, and a line through these also passes through the center of the required circle. Using as a center the intersection of these lines at X and taking a radius equal to AX, BX, or CX, the circle may be drawn.



The center of a circle circumscribing a triangle is the intersection of three perpendiculars bisecting the sides of the latter; the intersection of three lines bisecting the angles of a triangle is the center of an inscribed circle.

Two triangles are said to be *similar* when the angles of one are respectively equal to the angles of the other. By drawing DE parallel to BC, the portion ADE is similar to the whole triangle ABC.

The sides of similar triangles respectively opposite equal angles are called *homologous* sides; these are proportional:

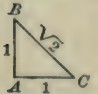


$$AD : AB :: AE : AC$$
$$AD : AB :: DE : BC$$

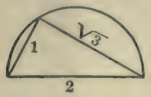
That is, if AD is $\frac{2}{5}$ of AB, AE is $\frac{2}{5}$ of AC and DE is $\frac{2}{5}$ of BC.

The areas of similar triangles are proportional to the squares of the homologous sides. If AB is $2\frac{1}{2}$ times AD, the area of the large triangle is $6\frac{1}{4}$ times the area of the small one. This also holds true of all regular surfaces: squares, hexagons, circles, etc., a circle whose radius is double that of another having an area four times as great, etc.

If it is desired to construct a circle, for instance, twice as large as that of a circle of 1-inch radius, the problem consists in drawing a radius measuring $\sqrt{2}$ inches. This is done by erecting a 1-inch perpendicular at the end of a 1-inch line and drawing the hypotenuse. The latter gives the length of the radius of the required circle.



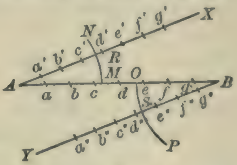
The side of a regular polygon with an area twice that of a given polygon is obtained by constructing an isosceles right triangle whose equal sides are equal to those of the given polygon. The hypotenuse indicates the length of the side of the required polygon.



By employing one side of a given regular polygon as the radius of a semicircle, and inscribing a right triangle having one side equal to the radius, the remaining side will be the length of the side of the polygon having an area three times that of the first.

The problem of dividing a line into equal parts is illustrated by the division of a given line AB into 7 equal portions:

From A a line AX is drawn at any convenient angle to AB. With A as a center, an arc MN is drawn, and with B as a center, using the same radius, the arc OP is drawn. The distance between M and R is next measured by the compasses, and OS laid off equal to MR. The line BY is then drawn through S, thus making the angle ABY equal to the angle BAX. The compasses are now used to mark off 7 equal divisions on AX, beginning at A, and 7 of the same length on BY, beginning at B, the former being shown at a', b', c', etc., to g', and the latter at g'', f'', e'', etc., to a''. By placing the ruler successively on a' and a'', on b' and b'', c' and c'', etc., the points a, b, c, d, etc., are located, which divide AB into 7 equal parts.



When AB is to be divided into any other number of equal parts, say 3, 5, 9, for instance, 3, 5, 9 equal divisions are laid off on AX and BY, respectively, and the proper intersections made.

The ratio between the homologous sides of similar triangles is employed in calculating heights and distances.

The height of a tree, steeple, etc., may be ascertained by comparing the length of its shadow with that of one cast at the same time by a stick. The height of the tree AB, for instance, is obtained by the following proportion, in which ab represents a stick, the respective shadows being denoted by BC and bc.

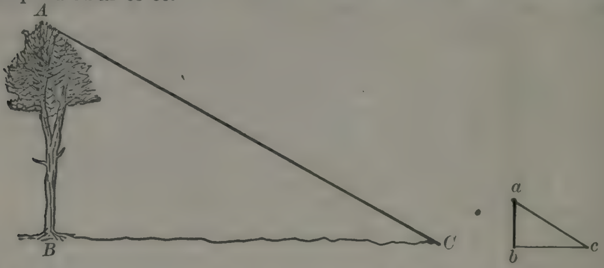
$$cb : ab :: CB : AB$$

When $cb = 7$ ft., $CB = 42$ ft., $ab = 5$ ft., and the proportion takes this form:

$$7 : 5 :: 42 : AB$$

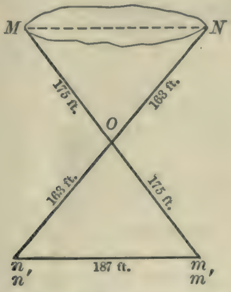
from which AB is found to be 30 feet.

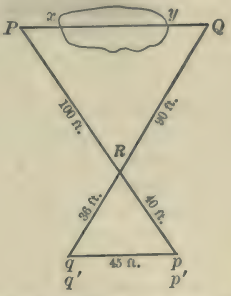
The shadows taken are those cast at the same time and ab must be parallel to AB; that is, when AB is perpendicular to BC, ab must be perpendicular to bc.



Long lines are measured by wire tapes, or by chains. The regular surveyor's chain is 4 rods (66 feet) long, divided into 100 links. The length of a line is stated in chains and a decimal; 4.83 chains, for instance. Another type is 50 feet long, also divided into 100 links; lengths in this case are given in feet, each link measuring ½ foot. In measuring the distance between two points that are not connected by a road, a fence, or the like, a pole is erected at each extremity as a guide. The bearer of the front end of the chain carries a dozen wire pegs, the top of each containing a bend to which is tied a piece of muslin to make the peg conspicuous. The bearer walks forward the length of the chain, and the rear man places the other end at the starting-point and sights along the chain to determine that it is in a line with the distant pole, signaling to the front man to step to the left or right, as may be necessary. As soon as the first man is notified that he is on the line, he pulls the chain taut and sticks a peg in the ground at his end of the chain. When the rear man reaches this peg, the same procedure is followed, the front man sticking a second peg into the ground and the rear man picking up the first one. When the front man reaches the end of the line, he folds up the surplus portion of the chain and announces the number of links beyond the last peg. This added to the number of pegs held by the rear man gives the length of the line. During the progress of the work the front man also sights along the chain to see that it is in a line with the pole at the starting-point.

One method of ascertaining the distance between *M* and *N*, that determines the length of a pond, for instance, is to set off an equivalent distance whose length can be measured. In the illustration a pole is set up at *O*, and its distance from *M* is found to be 175 feet. On a line with *M* and *O* a pole is set up at *m′*, and on this line a peg is placed at *m*, 175 feet from *O*, the latter being midway between *M* and *m*, on the line joining these two points. On a line with *N* and *O* a pole is set up at *n′*; the distance between *N* and *O* is then found, 163 feet, and 163 feet from O, on a line joining *O* and *n′*, a second peg is driven at *n*. The distance between *m* and *n* is now measured and shows the distance between *M* and *N*, in this case 187 feet.





The same result would be obtained by making *Om* 163 ft. and *On* 175 ft.

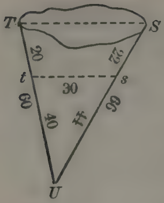
When it is not convenient to prolong *PR* and *QR* very far, the extensions *Rp* and *Rq* may be made of any length, care however being taken to have the ratio of *Rp* to *RP* the same as that of *Rq* to *RQ*. In the illustration *RP* is 100 ft., and *Rp* 40 ft., or $\frac{2}{5}$ of the former. If *RQ* measures 90 ft., *Rq* must be laid off $\frac{2}{5}$ of 90 ft., or 36 ft. The length of *qp* is found to measure 45 ft., which is $\frac{2}{5}$ of the length of *PQ*.

The value of *PQ* is indicated by either of the following proportions:

$$pR\ (40) : PR\ (100) :: qp\ (45) : PQ$$
$$qR\ (36) : QR\ \ (90) :: qp\ (45) : PQ$$

When *P* and *Q* are not on the edges of the pond, the sum of the respective lengths of *Px* and *yQ* is deducted to find the length of *xy*.

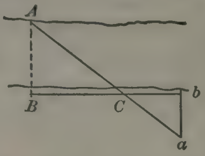
It will be noticed that *qR* and *QR* are homologous sides, as are *pR* and *PR*.

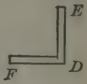


The line *st* may fall within the triangle. In the given example *TU* measures 60 yards and *SU* 66 yards. A stake is set at *t*, 40 yards from *U*, *Ut* being $\frac{2}{3}$ of the length of *UT*. Another stake is placed $\frac{2}{3}$ of 66 yards from *U*, on the side *US*. The length of *ts* will then be $\frac{2}{3}$ of the length of *TS*.

$$Ut\ (40) : UT\ (60) :: ts\ (30) : TS$$

To determine the width of a stream, that is, the length of *AB*, a line *Bb* perpendicular to *AB* is marked off by poles at *B* and *b*. A pole is then set at *a*, indicating a line *ab* perpendicular to *bB*. A pole is finally placed at *C*, the point of intersection of *Aa* and *Bb*.

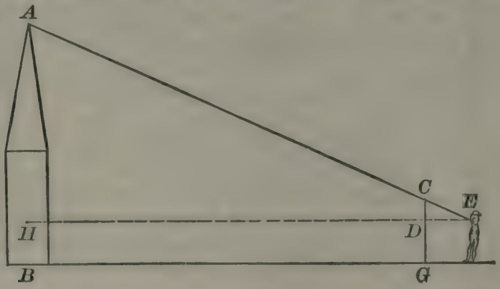




To make *Bb* perpendicular to *AB*, and *ab* perpendicular to *bB*, two strips of wood joined at right angles are fastened in a horizontal position on a vertical staff. When *D* is placed vertically above *B*, and *DE* takes the direction of *BA*, a person sighting along *FD* can indicate the location of the stake at *b*. The same procedure at *b* gives the point *a*. To locate *C*, one person stands at *a* and a second at *b* or *B*, the former directing a third person along the line *aA* until the observer at *b* (or *B*) announces that the third is on the line *bB;* the latter then places the pole at this point, *C*. From the measurements of *BC*, *Cb*, and *ba* the length of *BA* is calculated.

$$bC : BC :: ab : AB$$

The height of a tree or of a building may be ascertained by employing a vertical staff, *CG*, or a convenient fence. In the illustration the



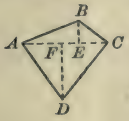
observer measures the distance (ED) he must be from CG, in order to bring the top of the object into view, the distance from the height of his eye (D) to C, the top of the fence, also the distance EH. From the following proportion the height of A above H is calculated:

$$ED : EH :: CD : AH$$

By adding to AH the distance from his eye to the ground, DG (or HB), the height of the building (AB) is found.

When a pole is used at CG, it must be set vertically, which can be done by means of a plumb line.

The two perpendicular strips employed to lay off a line at right angles to a given line are useful in locating the point on the base of a triangle from which its altitude is measured. The area as ascertained from the respective lengths of the sides may be used to test the result obtained by means of the base and the altitude. The same means may be employed in locating the perpendiculars let fall upon the diagonal of a quadrilateral from the opposite angles.



CHAPTER III

THE METRIC SYSTEM

Many courses of study provide for a brief treatment of the metric system in the later years of the elementary school. Although teachers should make themselves acquainted with the system and its advantages, they should not occupy the time of their pupils by extending instruction in it beyond the details specifically required.

It may be well to introduce the subject by means of a few questions as to the relation of the different units in our system of weights and measures. These should bring out the fact that a liquid gallon contains 231 cubic inches, that a bushel contains 2150.4 cubic inches, that a cubic foot of water weighs $62\frac{1}{2}$ pounds, etc. If the pupils have become acquainted with the use of troy weight, attention may be directed to the fact that its ounce and pound differ from those of avoirdupois weight, and also that the former has other units, the pennyweight and the grain, and that there is still another kind used by druggists with its scruples and drams, not to speak of the carat used in weighing precious stones. The complexity of our system is also shown by the difference in the size of the liquid and the dry quart.

A few questions will bring out the variety of ratios between the successive units of the different tables; the 16 ounces to the pound and the 2000 or 2240 pounds to the ton in one case; the 12 inches to the foot, 3 feet to the yard, and $5\frac{1}{2}$ yards to the rod in another, etc.

The treatment of the metric system should consist chiefly in the reading of the matter contained in the text-book, with the discussion of the various features that tend to constitute its remarkable simplicity to those who employ it exclusively. The decimal scale is brought out when the table of length is read, and the fact that multiples or subdivisions of the meter are employed in each of the other tables.

Another feature is the ease with which all reductions are made. While the strange names tend to confuse our pupils, they can readily appreciate the simplicity of a method that employs only a power of 10 as a multiplier or a divisor.

The connection between the table of length and that of capacity also shows how much easier it is for a European pupil to ascertain the quantity of grain or of oil a bin or a tank will hold, when the result is required in liters in each case, there being 1000 liters in a cubic meter.

The unit in the table of weight is the gram; 1000 of these, a kilogram, being the weight of a liter of water. The kilogram is generally called a kilo, especially in this country.

Although it is inadvisable to burden the pupils with details not contained in their text-books, it may be useful for the teacher to be acquainted with the foreign method of writing the abbreviations. In France the use of the capital letters D, H, and K to denote the prefixes deca, hecto, and kilo, respectively, has been discontinued; da, h, and k being substituted. The decameter, decaliter, decagram, etc., are indicated by dam, dal, and dag, respectively, to distinguish them from the decimeter, deciliter, and the decigram. All the other abbreviations use small letters exclusively except the myriameter (10,000 meters), etc., which is written Mm to avoid confusing it with the millimeter. The prefix M is, however, seldom used.

No period is used in connection with these abbreviations, which are generally written above the line, and are located after the whole number and before the decimal. Thus 24 meters and 75 centimeters is indicated: $24.^{m}75$, except that a comma is used to denote the decimal point. The foregoing is read in the same way as would be $24.75 in this country, and not as 2 decameters 4 meters 7 decimeters 5 centimeters, just as we are not likely to read the other as 2 eagles 4 dollars 7 dimes 5 cents.

All of the units in each table are not in regular use, no more than are our units of dimes and eagles in the money table. Widths of boards, carpets, ribbons, etc., are generally stated orally in centimeters, such as 10 centimeters, 50 centimeters, and even 100 centimeters, and may be written 10^{cm}, 50^{cm}, 100^{cm}, just as we write 10¢, 50¢, and even 100 cents. The usual way, however, to write them is: $0.^{m}10$, $0.^{m}50$, $1.^{m}00$, corresponding, respectively, with $0.10, $0.50, $1.00.

To express the length of a road, for instance, measuring 16,480 meters, the kilometer is taken as the unit, the whole length being stated as 16 kilometers 480 meters and written $16.^{km}480$. This method emphasizes the important feature, 16 kilometers, while not overlooking the meters.

In Germany the abbreviations are generally denoted by letters in italics written on the line and placed after the decimal. The terminal ciphers in the latter are also omitted, as a rule, the expression in the last paragraph being written 16.48 *km*, although it is retained in writing money, such as 4.80 *M*, for instance.

In measures of surfaces, large areas are expressed in ares of 100 square meters each. An are is the equivalent, therefore, of a square having sides of 1 decameter. A field 84 meters long and 75 meters wide would contain 6300 square meters, or 63 ares, the use of the larger unit being a convenience in stating the area in smaller numbers. The change from square meters to ares is made by pointing off two decimal places, or the dimensions given above may be stated in decameters, 8.4 × 7.5, the product giving the number of ares. The United States and England labor under the disadvantage of having no integral unit to denote the side of a square containing 1 acre.

The volume of a pile of wood is stated in steres, another and shorter expression for cubic meters. A large weight is given in metric tons of 1000 kilos each. No allusion to these units should be made if they are not mentioned in the text-book.

In some schools the pupils use rulers having metric divisions on one side and inch divisions on the other. These rules cost the same as those of the ordinary type and they are useful in laying off and measuring lines without using fractions. Pupils employing these rulers acquire a familiarity with the decimeter, centimeter, and millimeter. The five-cent piece affords an illustration of some units, its diameter being 2 centimeters, its thickness 2 millimeters, and its weight 5 grams.

While some exercises employing metric units and their equivalents in our system may be used in any grade when all of the necessary conditions are stated in each example, too much time of the older pupils should not be given to this kind of work.

Exercises in changing from metric units to ours should be limited largely to sight and oral exercises, in which approximate equivalents are used. In changing meters to inches, yards, or feet, the former may be taken as 40 inches, $1\frac{1}{12}$ yards, or $3\frac{1}{4}$ feet. The mental reduction of $\frac{5}{8}$ mile to 200 rods, 1100 yards, 3300 feet, 39,600 inches, and the comparison of the latter with 39,370 inches, will show a pupil that 1000 meters, a kilometer, is nearly $\frac{5}{8}$ mile.

The relation of the liter and the quart may be ascertained mentally by considering the former as a cube having sides of a little less than 4 inches ($\frac{1}{10}$ of 40 inches), making its volume something less than 64 cubic inches. As the liquid quart, of $\frac{1}{4}$ of 231 cubic inches, contains $57\frac{3}{4}$ cubic inches, it is a little smaller than the liter. The weight of a pint of water being about 1 pound, a liter of water, which weighs a kilogram, weighs over 2 pounds.

Written problems should first be treated as a class exercise, each problem being read and an approximate answer obtained before the pupils are expected to solve the problem at their seats. This procedure affords an opportunity for the teacher to note the points in the preliminary work that need further attention.

The numerous attempts to legalize the use of the metric system in England and the United States have met with no success. The former country is beginning to appreciate the value of a decimal currency; but in both, the enormous loss to manufacturers that would be made necessary by the change is an obstacle that continually grows greater. Russia holds out against Greenwich time, uniform currency, etc., but for other reasons. France, Belgium, Spain, Switzerland, Italy, and Greece have the same monetary unit, equivalent to 19.3 cents of our money, with some differences in name. Although Germany, in common with these countries, employs the metric system and uses time based upon that of England, it has a different currency. France has only just accepted Greenwich time. In all countries, however, the metric system, including the centigrade thermometer, is used in scientific work.

Even our different states do not have entire uniformity as to the legal weights that constitute a bushel of the various grains, etc., although agreeing as to the necessity of weighing instead of measuring. In some cities small dealers are required by law to weigh out potatoes to a customer buying a peck, the use of the measure for this purpose being forbidden.

The reasons underlying many of these usages are easily understood by pupils, and courses of study occasionally suggest that questions be put to pupils to elicit the necessity of some of these regulations, such as the selling of eggs by weight instead of by count, of bread by the pound rather than by the loaf, etc. When time affords, a very brief discussion of these topics in connection with work in denominate numbers is more valuable than long and tiresome calculations that are of no practical use.

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Author Walsh, John H.

Title Practical methods in Arithmetic.

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